A new financial calculus
or
the problem of computing hedging errors in incomplete markets

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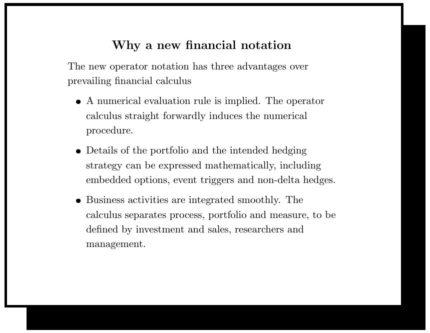
Stefan Dirnstorfer

2004-Feb-26

A new financial calculus

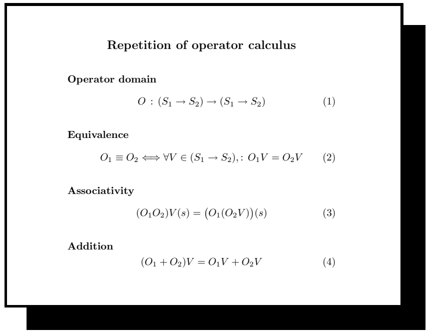


Why a new financial notation

The new operator notation has three advantages over prevailing financial calculus

- A numerical evaluation rule is implied. The operator calculus straight forwardly induces the numerical procedure.
- Details of the portfolio and the intended hedging strategy can be expressed mathematically, including embedded options, event triggers and non-delta hedges.
- Business activities are integrated smoothly. The calculus separates process, portfolio and measure, to be defined by investment and sales, researchers and management.

Why



Repetition of operator calculus

Operator domain

$$O : (S_1 \to S_2) \to (S_1 \to S_2) \quad (1)$$

Equivalence

$$O_1 \equiv O_2 \iff \forall V \in (S_1 \to S_2),: O_1V = O_2V \quad (2)$$

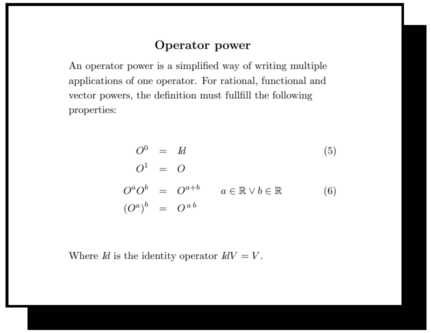
Associativity

$$(O_1O_2)V(s) = \big(O_1(O_2V)\big)(s) \quad (3)$$

Addition

$$(O_1 + O_2)V = O_1V + O_2V \quad (4)$$

Operators1



Operator power

An operator power is a simplified way of writing multiple applications of one operator. For rational, functional and vector powers, the definition must fullfill the following properties:

$$\begin{aligned} O^0 &= \mathit{Id} && (5)\\ O^1 &= O &&\\ O^aO^b &= O^{a+b} \qquad a \in \mathbb{R} \vee b \in \mathbb{R} && (6)\\ (O^a)^b &= O^{a\cdot b} && \end{aligned}$$

Where Id is the identity operator $\mathit{Id}V = V$.

Operators2

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Operators2

A new financial calculus

Introduction

Financial models

Implementation

Financial Model

A financial model is largely set up by three components: the process, the portfolio and a measure.

- The time **process** is a stochastic description of the path an observable variable can take.
- A **portfolio** describes a set of investments or the room for further trading activity.
- A **measure** is a computable property of the model. (E.g. expectations, distributions and parameters)

What is it?

Notation

Suppose process and portfolio are each represented by an operator named Θ and A. The measure is a function V, such that

$$(\Theta A)^M V(s) \tag{7}$$

becomes a unique and easy to evaluate answer to the question: What is the measure V of portfolio A under process Θ over a horizon of length M, given a current economic state s.

Notation

Process

Portfolio

Measure

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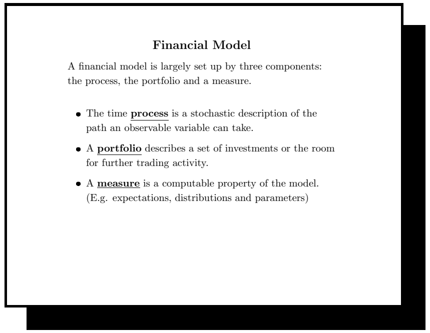

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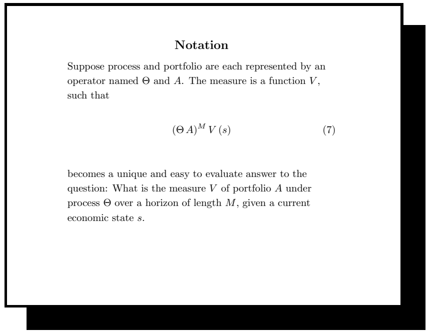

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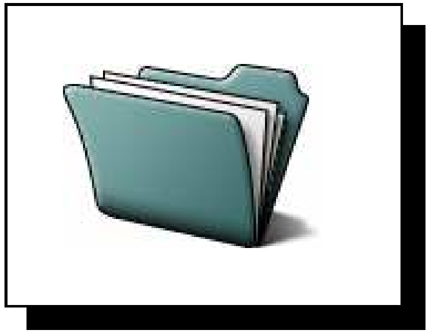
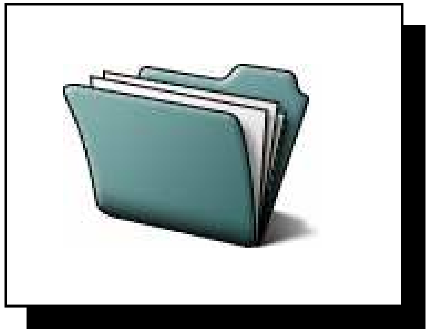
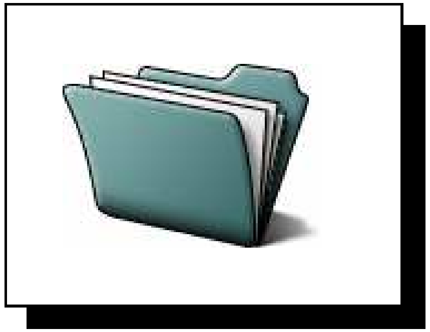


Process



Portfolio



Measure

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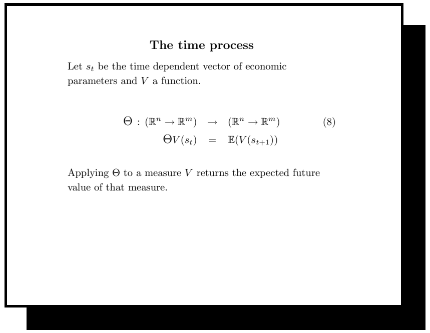
- Introduction
- ⊟ Financial models
  - ⊞ Process
  - ⊞ Portfolio
  - ⊞ Measure
- ⊞ Implementation



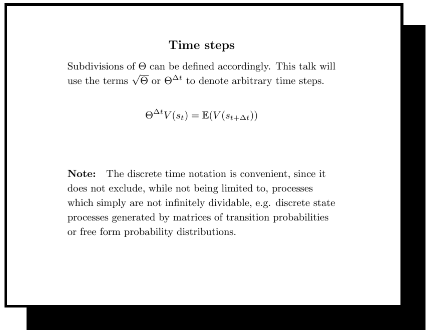
The time process

Let s_t be the time dependent vector of economic parameters and V a function.

$$\begin{aligned} \Theta : (\mathbb{R}^n \to \mathbb{R}^m) &\to (\mathbb{R}^n \to \mathbb{R}^m) \\ \Theta V(s_t) &= \mathbb{E}(V(s_{t+1})) \end{aligned} \tag{8}$$

Applying Θ to a measure V returns the expected future value of that measure.

Θ operator



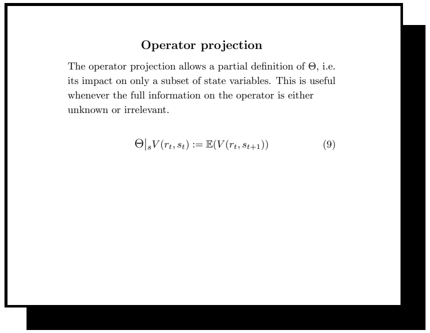
Time steps

Subdivisions of Θ can be defined accordingly. This talk will use the terms $\sqrt{\Theta}$ or $\Theta^{\Delta t}$ to denote arbitrary time steps.

$$\Theta^{\Delta t} V(s_t) = \mathbb{E}(V(s_{t+\Delta t}))$$

Note: The discrete time notation is convenient, since it does not exclude, while not being limited to, processes which simply are not infinitely dividable, e.g. discrete state processes generated by matrices of transition probabilities or free form probability distributions.

Time steps

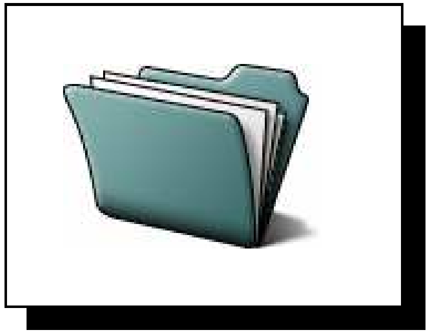


Operator projection

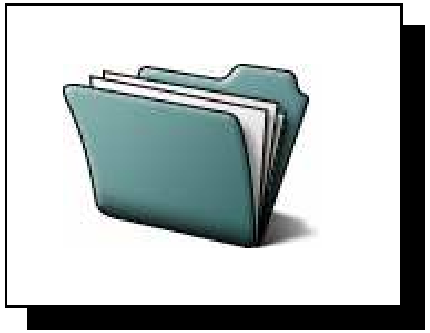
The operator projection allows a partial definition of Θ, i.e. its impact on only a subset of state variables. This is useful whenever the full information on the operator is either unknown or irrelevant.

$$\Theta\big|_s V(r_t, s_t) := \mathbb{E}(V(r_t, s_{t+1})) \tag{9}$$

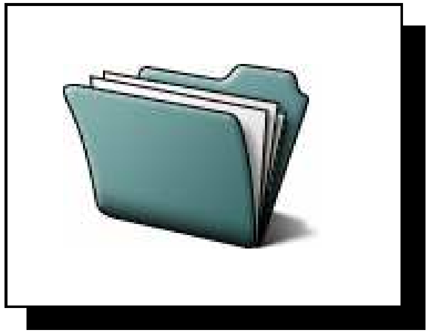
Projection



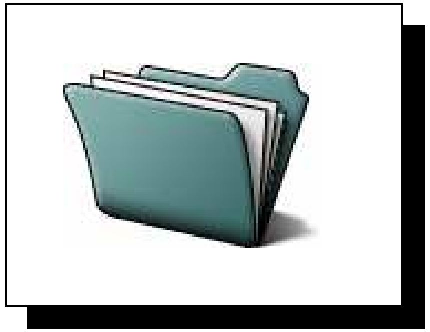
Shift



Blur



Event



Copula

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Projection

Shift

Blur

Event

Copula

The shift operator

The operator T_x transforms a measure V by a deterministic growth in variable x.

Definition The operator definition declares the T operator by the transport equation with velocity ν over a unit time interval.

$$\frac{dT_x^{\nu\epsilon}}{d\epsilon}V = \nu\frac{d}{dx}V \qquad (10)$$

The variable ϵ plays the role of the time variable in standard PDE theory.

Example: Interest rate account

The following example models a cash account c that receives an interest rate dependend coupon cr_t in each period.

$$\begin{aligned} T_t V(t) &= V(t+1) \\ T_c^{cr} V(c) &= V(e^r c) \end{aligned} \tag{11}$$

The shift $T_c^{cr_t}$ pays the time dependend rate r_t and T_t increases a time variable t in each period.

$$\begin{aligned} (T_c^{cr_t} T_t)^M V(c) &= (T_c^{cr_t} T_t)^{M-1} V(e^{r_t} c) \\ &= (T_c^{cr_t} T_t)^{M-2} V(e^{r_t} e^{r_{t+1}} c) \\ \cdots &= V\left(c \prod_{p=0}^{M-1} e^{r_{t+p}}\right) \end{aligned} \tag{12}$$

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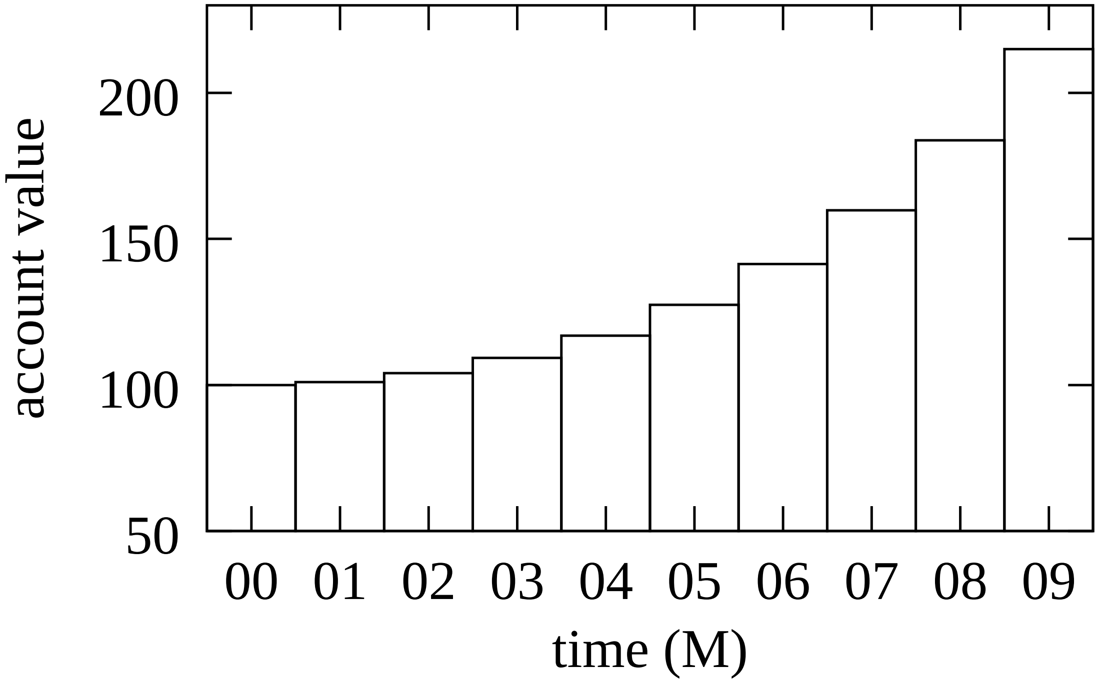
Plotting the shift

Let the time process be:

$$\Theta = T_c^{cr_t} T_t \tag{13}$$

with

$$r_t = 1\% + 2\% t \tag{14}$$



Plotted function: $f(M) = \Theta^M c(100, 0)$

The blur operator

As soon as uncertainty enters the model we need a diffusion operator. The blur operator B solves the heat equation that has Brownian motion as source of randomness.

Definition The operator B solves the heat equation with diffusion coefficient $\sigma^2/2$. The size of σ determines the amount of uncertainty and was named **volatility** in financial terminology.

$$\frac{dB_x^{\sigma^2\epsilon}V}{d\epsilon} = \frac{\sigma^2}{2}\frac{d^2}{dx^2}V \tag{15}$$

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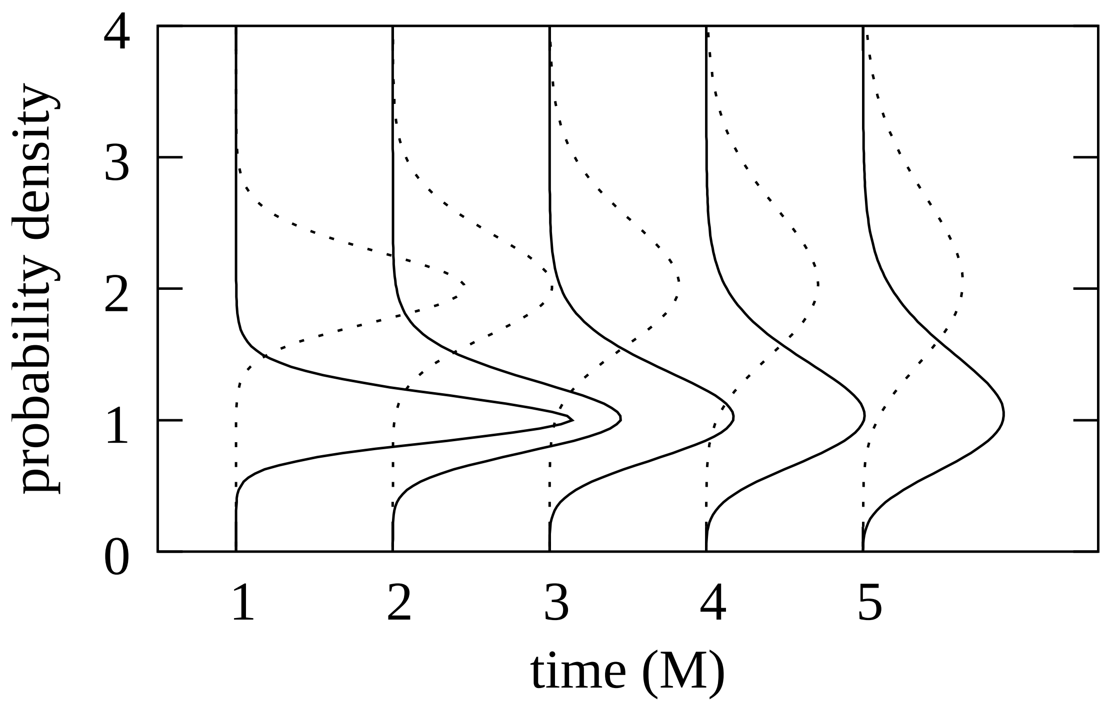
Bluring a delta function

Let the time process be:

$$\Theta = B_S^{(\sigma S)^2} \tag{16}$$

with $\sigma = 20\%$



$V(x) = \delta(x-1),\ \delta(x-2)$

Plotted function: $f(M, x) = \Theta^M V(x)$

Black & Scholes

The Black-Scholes model has Brownian motion as driving process, a european option in the portfolio and measures the fair premium for the option. Using the shift and blur operator, the time process Θ_{BS} of this famous model can be rewritten.

$$\Theta_{BS} = e^{-r}\, T^{rS}\, B^{(\sigma S)^2} \tag{17}$$

The process is built from three **individual operations**: discounting e^{-r}, stock price drift T^{rS} and stock price diffusion $B^{(\sigma S)^2}$.

Note: All operators commute for constant σ and r

Heath-Jarrow-Morton

The HJM model propagates a forward rate curve $f(M)+r$, whereas r is the rate change (initially 0).

$$\Theta_{HJM} = B_r^{\sigma^2} T_r^{\sigma\sigma^*} T_t \tag{18}$$

with

$$\sigma^* = \int_{t+\frac{1}{2}}^{M} \sigma dt' \tag{19}$$

whereas σ may depend on r, t and M.

Note 1: The integration starts with $t+\frac{1}{2}$ instead of t due to the discrete time step.

Note 2: r is a dimension and not a variable. Non parallel curve shifts are possible: $T_r^M V(1+r) = V(1+M+r)$

Events

If two scenarios O_1 and O_2 occur with probabilities p and $1-p$, their values can be combined linearly in the time process, since the expectation value $\mathbb{E}$, and therefore Θ, is linear

$$\begin{tikzcd}[column sep=large, row sep=small] & O_1 \\ \cdot \arrow[ur, dash, "p"] \arrow[dr, dash, "1-p"'] & \\ & O_2 \end{tikzcd} \quad := p\,O_1 + (1-p)O_2 \tag{20}$$

Copulas

After the processes for individual economic variables are determined there still remain questions about their joint behavior. When we know the projections

$$\Theta|_x$$

and

$$\Theta|_y$$

what constraints does this put on the joint process

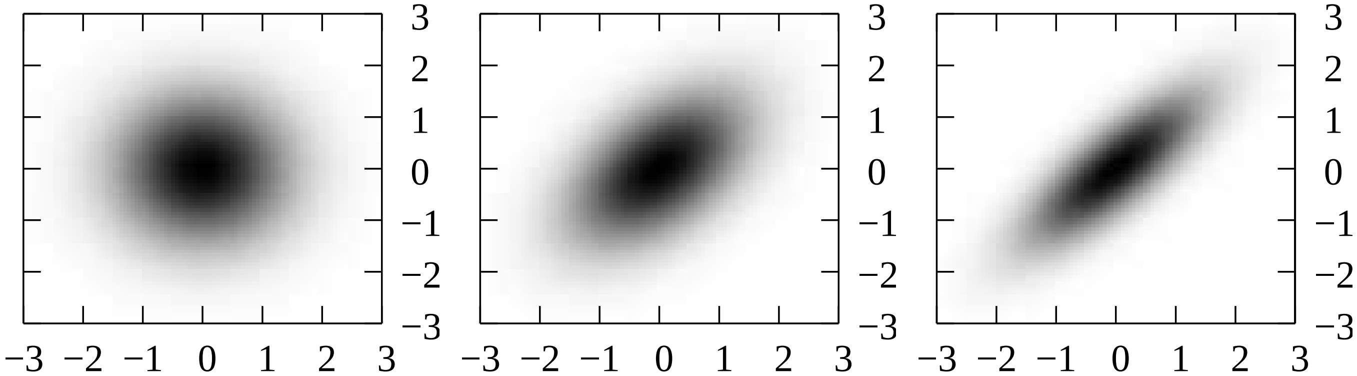
$$\Theta|_{x,y} \; ?$$

Correlation

The correlated diffusion can either be defined by a new PDE operator or on a transformed domain:

$$\Theta_{corr} = \underbrace{T_x^{-y} B_y^{\rho} T_x^{y}}_{\text{1-correlated}} \; B_x^{1-\rho} B_y^{1-\rho} \tag{21}$$

Correlations: 0, 0.5 and 0.8



Convolution

Using the T operator we can formulate the convolution with a probability distribution Φ:

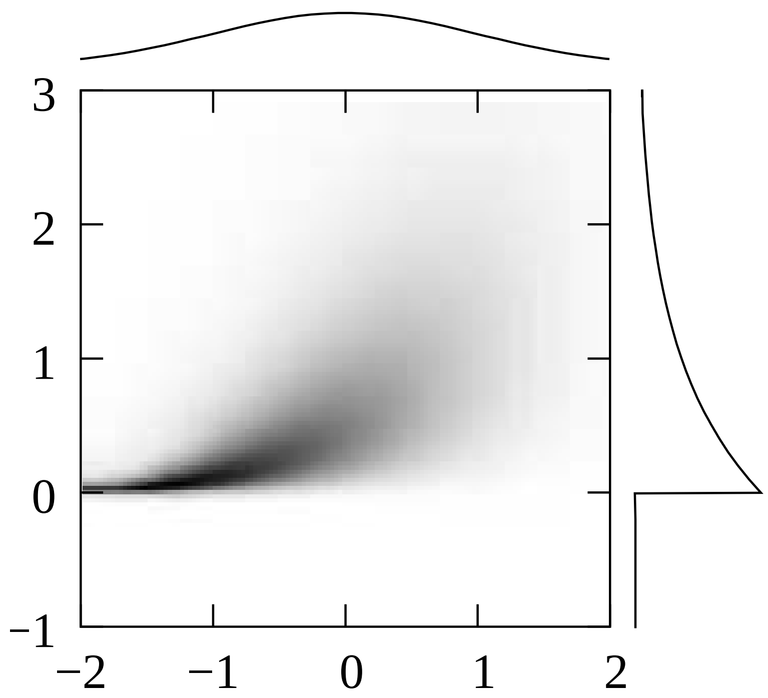
$$V * \Phi'(x) = \int_0^1 T_x^{-\Phi^{-1}(u)} V du \tag{22}$$

which is equivalent to the standard form:

$$\begin{aligned} \int_0^1 T_x^{-\Phi^{-1}(u)} V(x) du &= \int_0^1 V(x - \Phi^{-1}(u)) du \\ &= \int_{-\infty}^{\infty} V(x-u)\Phi'(u) du \end{aligned}$$

Example: The Clayton copula

$$\int_0^1 \int_0^1 \left(\frac{d^2}{du\,dv} C(u,v) \right) T_x^{-\Phi_1^{-1}(u)} T_y^{-\Phi_2^{-1}(v)} V\,du\,dv \tag{23}$$



$\Phi_1(x) = N(x)$
$\Phi_2(x) = 1 - e^{-x}$
$C(x,y) = exp((-ln(x))^\beta + (-ln(y))^\beta)^{1/\beta}$
$\beta = \frac{1}{2}$

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Shift

Blur

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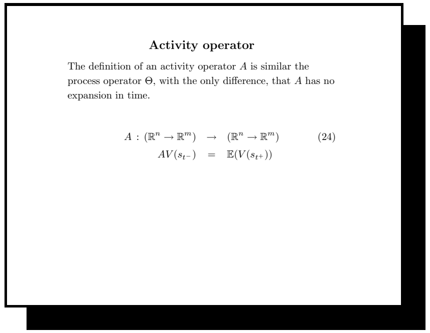
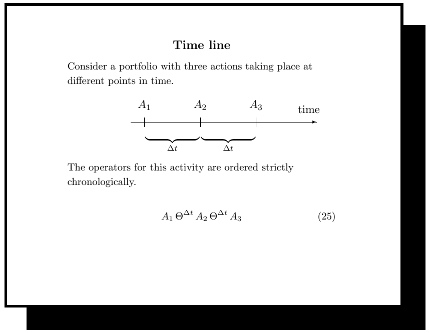


Activity operator

The definition of an activity operator A is similar the process operator Θ, with the only difference, that A has no expansion in time.

$$\begin{aligned} A : (\mathbb{R}^n \rightarrow \mathbb{R}^m) &\rightarrow (\mathbb{R}^n \rightarrow \mathbb{R}^m) \qquad (24) \\ AV(s_{t-}) &= \mathbb{E}(V(s_{t+})) \end{aligned}$$

Action operator



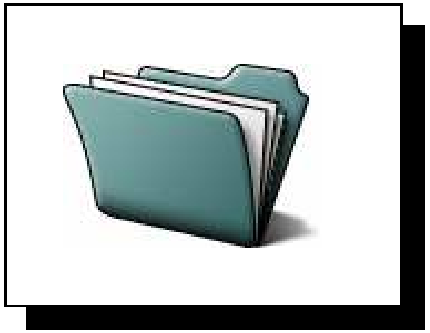
Time line

Consider a portfolio with three actions taking place at different points in time.

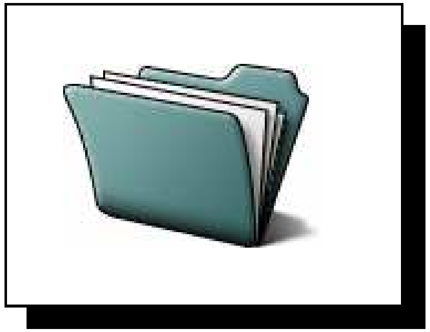
The operators for this activity are ordered strictly chronologically.

$$A_1 \, \Theta^{\Delta t} \, A_2 \, \Theta^{\Delta t} \, A_3 \qquad (25)$$

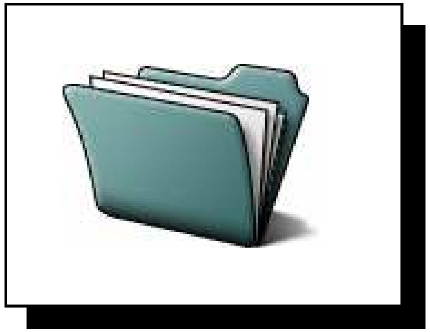
Time line



Transfer



Options



Strategies

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$$\begin{aligned} A : (\mathbb{R}^n \longrightarrow \mathbb{R}^m) &\longrightarrow (\mathbb{R}^n \longrightarrow \mathbb{R}^m) \\ AV(s_{t^-}) &= \mathbb{E}(V(s_{t^+})) \end{aligned} \tag{24}$$

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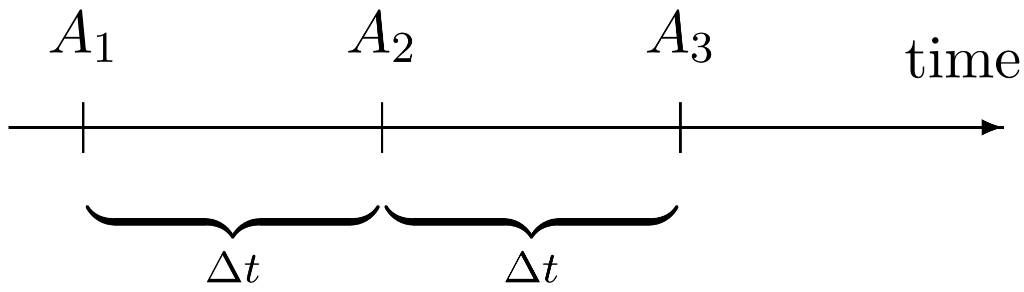

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    - Transfer
    - Options
    - Strategy
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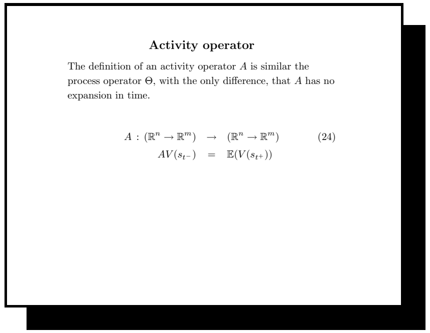

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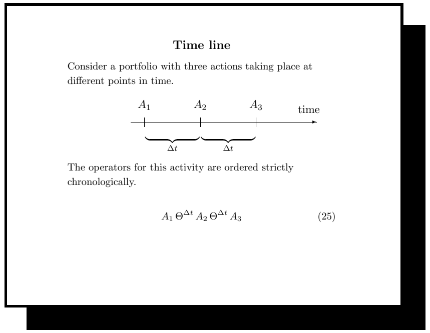
$$\begin{aligned} A : (\mathbb{R}^n \rightarrow \mathbb{R}^m) &\rightarrow (\mathbb{R}^n \rightarrow \mathbb{R}^m) \\ AV(s_{t-}) &= \mathbb{E}(V(s_{t+})) \end{aligned} \tag{24}$$

Action operator



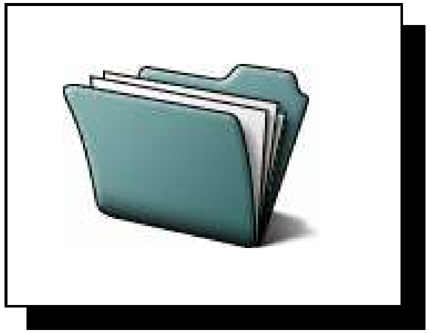
Time line

Consider a portfolio with three actions taking place at different points in time.

The operators for this activity are ordered strictly chronologically.

$$A_1 \, \Theta^{\Delta t} \, A_2 \, \Theta^{\Delta t} \, A_3 \tag{25}$$

Time line



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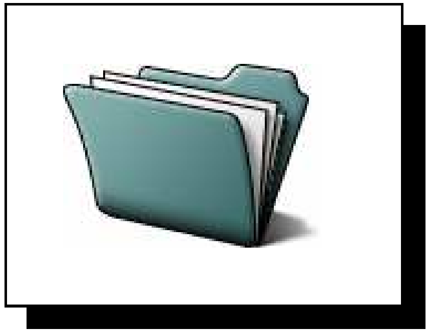

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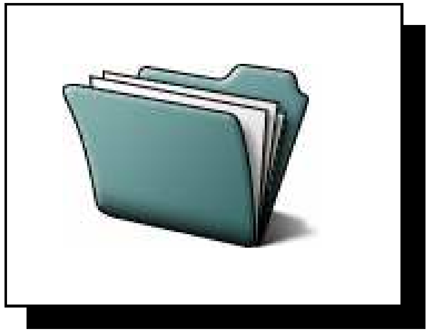


Strategies

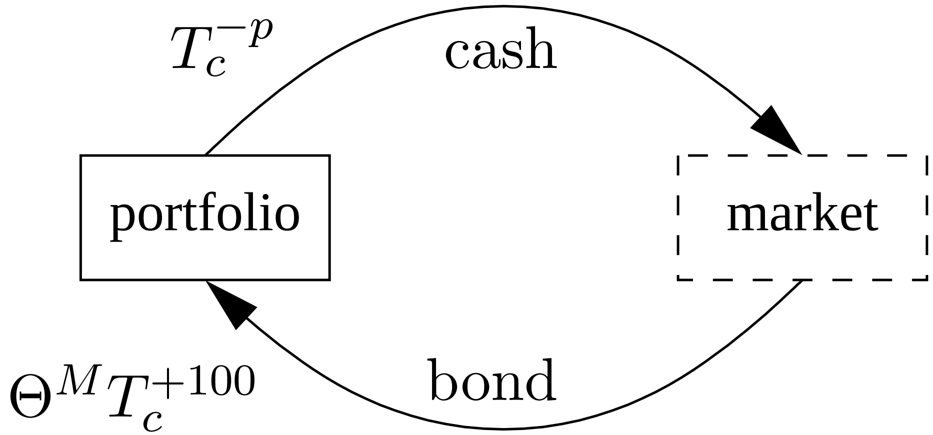
Transfer

A transfer of goods, assets or cash is performed by the T operator.

$$T_c^n \tag{26}$$

Whereas n is the number of units transfered into the account c.

A bond trade



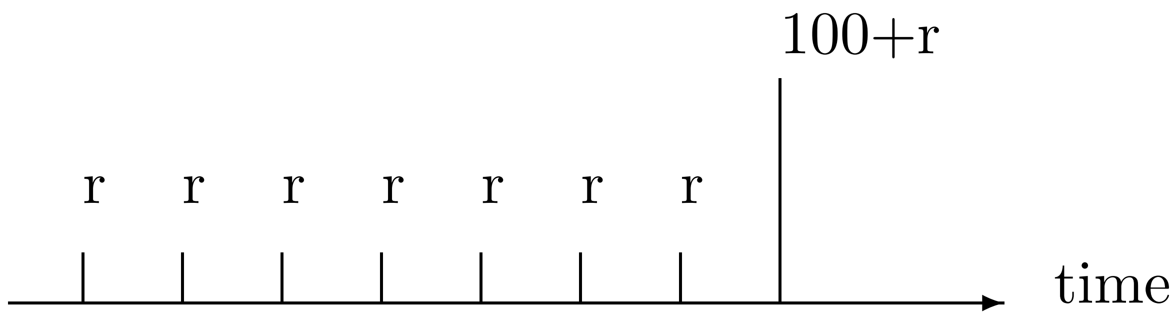
A bond trade First the bond's premium p is extracted from the account c. After M periods the bonds face value 100 is returned. A fair premium p with respect to measure V fulfills :

$$T_c^{-p} \Theta^M T_c^{100} V = 0 \tag{27}$$

A coupon bond

A coupon bond that pays rate r per period and a final redemption of 100.

$$\left(\Theta T^{\,r}\right)^{M} T^{100} \tag{28}$$



Options

An option is defined by the alternatives among which can be selected and by the entity that does select. The option values are denoted as V_1 and V_2. The deciding entity is chracterized by her utility operator U.

$$\begin{tikzcd} & V_1 \\ {} \arrow[ur, "UV_1>UV_2"] \arrow[dr, "UV_1\leq UV_2"'] & \\ & V_2 \end{tikzcd} \tag{29}$$

Time continuous option

An American option is an option with continuous execution right. The option requires a time process that can be infinitely subdivided and an option that is executable after each infinitely small time step.

Assume a utility operator U and a selectable option X. The process is defined by a limit of finite timesteps.

$$\lim_{\Delta t \to 0} \left(\Theta^{\Delta t} \begin{cases} \xrightarrow{U\,X > U\,.} X \\ \xrightarrow{U\,X \leq U\,.} . \end{cases} \right)^{\frac{M}{\Delta t}} \tag{30}$$

Space continuous option

Assume an action A that can be executed arbitrarily often. Let $A^\star$ be the optimal execution of A with respect to utility U:

$$A^\star = A^{x^*} \tag{31}$$

Whereas

$$x^* = \operatorname{argmax} U A^{x^*} \tag{32}$$

Example

Consider a portfolio manager who can buy an arbitrary amount of stocks at price S. The money is extracted from account c, while the stock is added to the portfolio h.

The optimal investment is:

$$(T_c^{-S}\, T_h)^{\star} \tag{33}$$

Delta Hedge

Assume two state variables c for the amount of cash in our account and h, the number of stock in our depot.

In order to hedge a trading activity A, consider a counter investment A_H that buys $-\Delta$ stocks.

$$A_H V = \left(T_c^{-S} T_h\right)^{-\Delta} V \tag{34}$$

Whereas Δ is the sensitivity of measure V to stock price changes

$$\Delta = \frac{d}{dS} V \tag{35}$$

Now the following time continuous strategy is risk free:

$$\lim_{\Delta t \to 0} \left(A_H \Theta^{\Delta t}\right)^{\frac{M}{\Delta t}} A \tag{36}$$

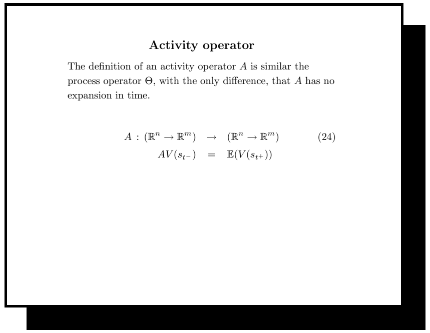
Best incomplete hedge

Assume that a hedging transaction A_H produces costs k relative to the traded volume $|\Delta|$.

$$A_H V = T^\star_\Delta \, T_c^{-k|\Delta|} \, (T_c^{-S} T_h)^\Delta V \tag{37}$$

With the optimization decision $T^\star$, the heding activity A_H optimizes the portfolio with respect to utility U.

A new financial calculus



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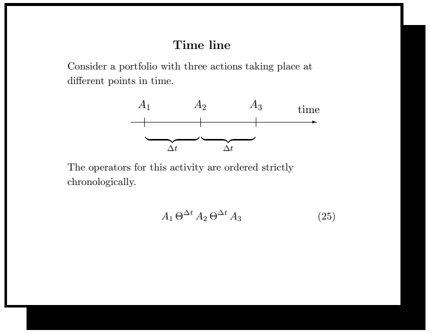

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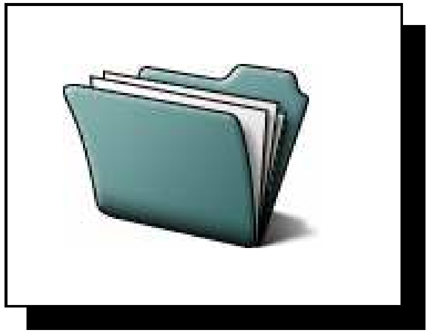

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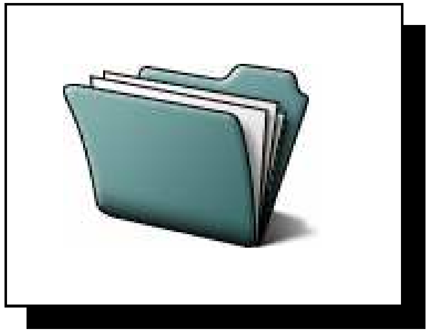

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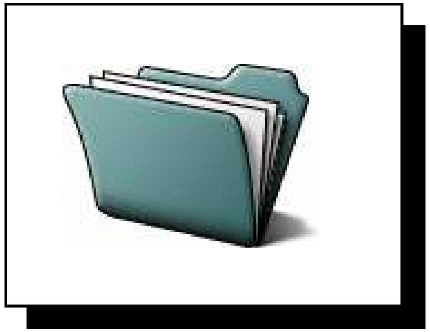
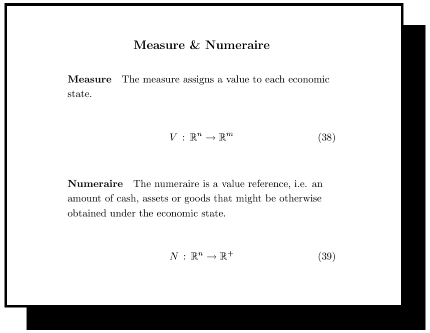


Strategies

A new financial calculus



Measure & Numeraire

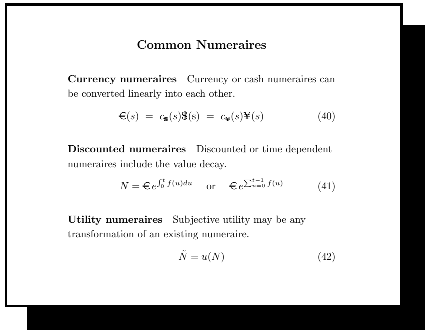
Measure The measure assigns a value to each economic state.

$$V : \mathbb{R}^n \to \mathbb{R}^m \tag{38}$$

Numeraire The numeraire is a value reference, i.e. an amount of cash, assets or goods that might be otherwise obtained under the economic state.

$$N : \mathbb{R}^n \to \mathbb{R}^+ \tag{39}$$

Numeraire



Common Numeraires

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$$€(s) = c_{\$}(s)\$(s) = c_{¥}(s)¥(s) \tag{40}$$

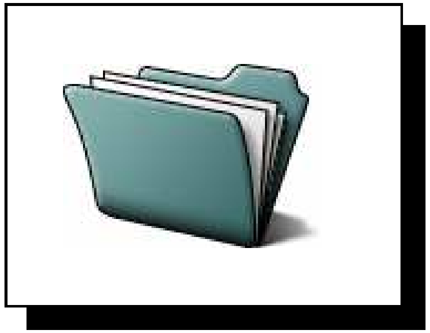
Discounted numeraires Discounted or time dependent numeraires include the value decay.

$$N = € e^{\int_0^t f(u)du} \quad \text{or} \quad € e^{\sum_{u=0}^{t-1} f(u)} \tag{41}$$

Utility numeraires Subjective utility may be any transformation of an existing numeraire.

$$\tilde{N} = u(N) \tag{42}$$

Numeraires



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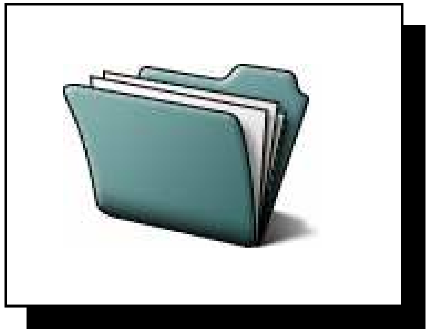

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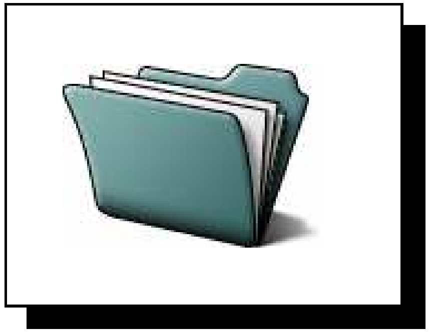

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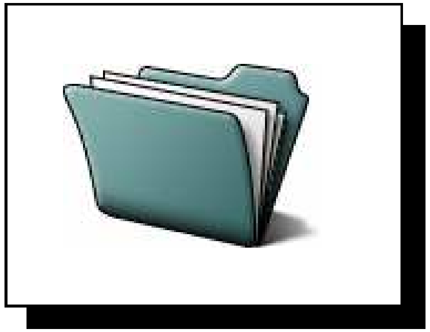

Parameter fitting

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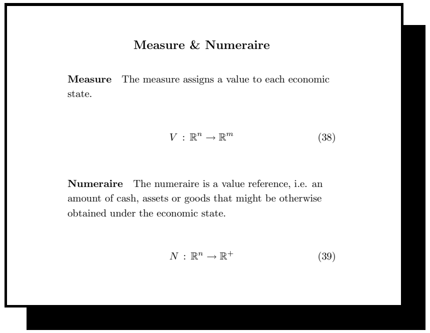
A new financial calculus



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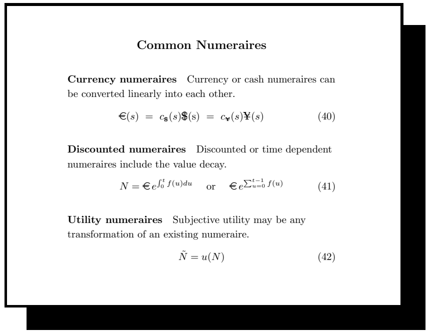

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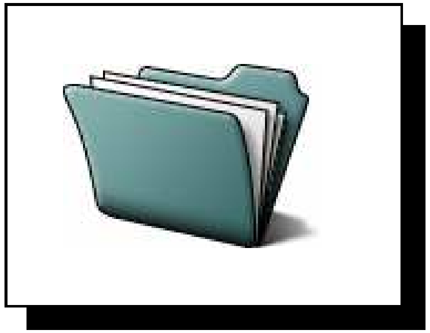

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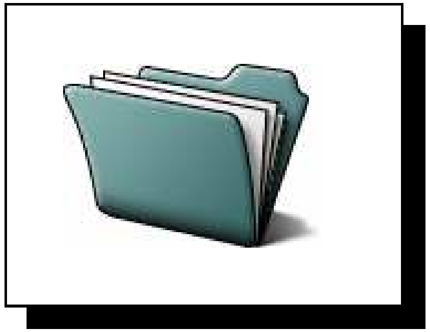

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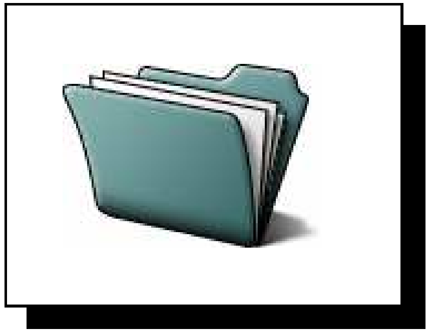

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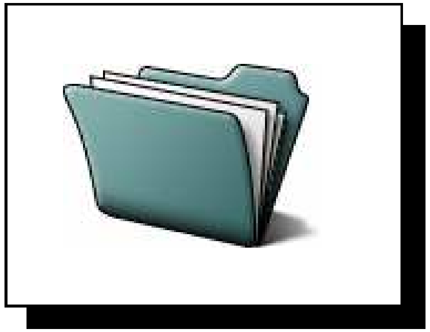


Parameter fitting

Sensitivity

Let the resulting information from a financial model be:

$$(\Theta A)^M \; V(s) \tag{43}$$

The sensitivity to can be computed for any influencing variable by differentiation.

$$\frac{d}{dp}\left(\Theta_p A_p\right)^M \; V_p(s_p) \tag{44}$$

The Greeks

Common characterisations of a portfolio:

$$\begin{aligned}
\Delta &= \frac{d}{dS}\,(\Theta A)^M N(s) \\
\Gamma &= \frac{d^2}{dS^2}(\Theta A)^M N(s) \\
\text{(Vega)}\ \upsilon &= \frac{d}{d\sigma}\,(\Theta A)^M N(s) \\
-\theta &= \frac{d}{d\tau}\,\Theta^\tau(\Theta A)^M N(s) \\
\rho &= \frac{d}{dr}\,(\Theta A)^M N(s)
\end{aligned} \tag{45}$$

A new financial calculus

Example

The process is Heath-Jarrow-Morton Θ_{HJM} with

$$\begin{aligned} \sigma &= \frac{1\%}{10\% + M^2}\sqrt{f(M)} \qquad (46) \\ f(M) &= 5\% + 0.5\% M + r \end{aligned}$$

The portfolio is a bond combined with a short bermudean option:

$$PV = \left(\Theta T_c^{7\text{€}}\right)^2 \left(\begin{array}{ll} \nearrow^{T^{105\text{€}}V \angle \cdot} & T_c^{105\text{€}}V \\ \searrow & \cdot \end{array} \quad \Theta T_c^{7\text{€}} \right)^8 T_c^{100\text{€}}V \qquad (47)$$

As measure, we want to know the discounted cash on our account:

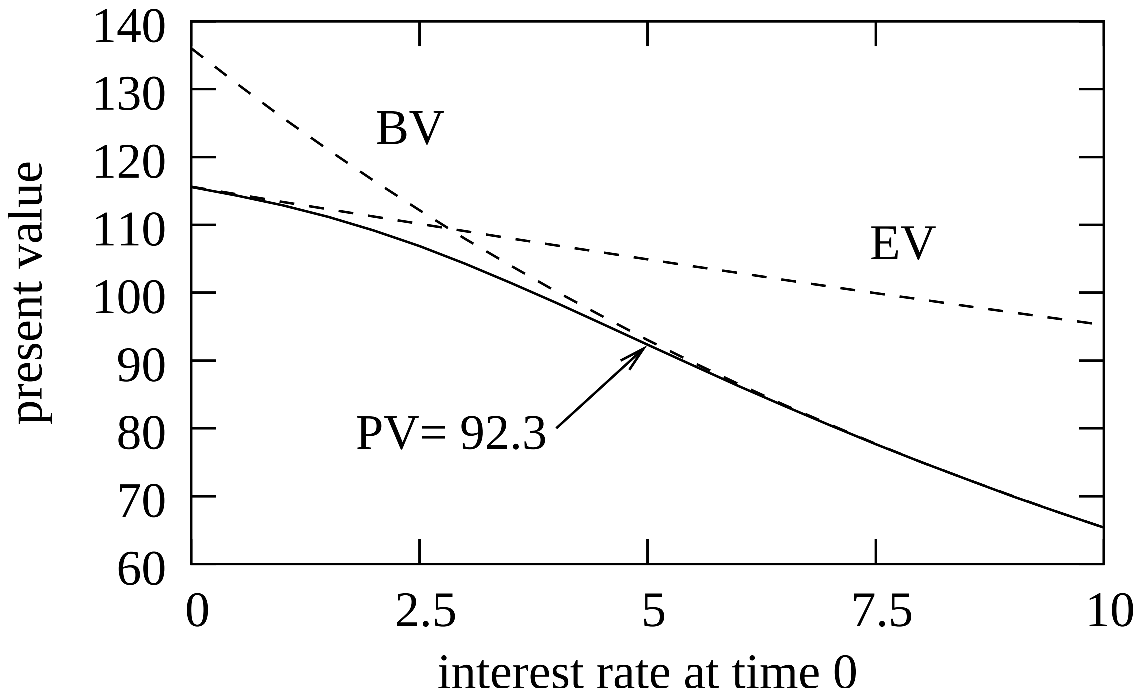
$$\text{€} = \exp\left(-\sum_{u=0}^{t-1} f(u)\right) \qquad (48)$$

Present value

Since the option is short (worst scenario is chosen) our portfolio value is striktly below all optional bonds.

Bond value: $B = (\Theta T^{7€})^{10}\, T^{100€}$

First execution value: $E = (\Theta T^{7€})^{2}\, T^{105€}$



Distribution

The probability distribution is a measure V, created from a numeraire N and a variable x.

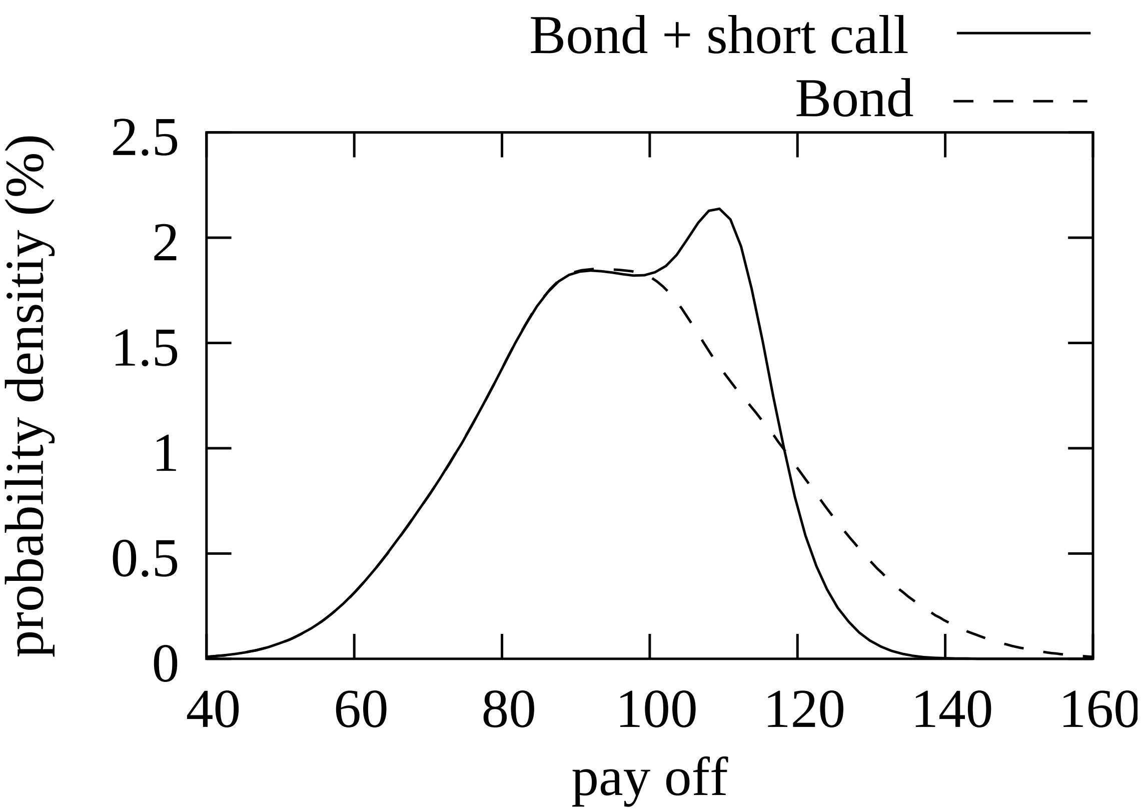
$$V = \begin{pmatrix} N \\ 1_{N>x} \end{pmatrix} \tag{49}$$

The utility operator U has to be adjusted to the new measure:

$$\tilde{U}V = UV_1 \tag{50}$$

Example

The option in the portfolio rules out very high revenues in favor of early payment just above the strike price.



Stop times

For stop time measure we use a stop condition C.

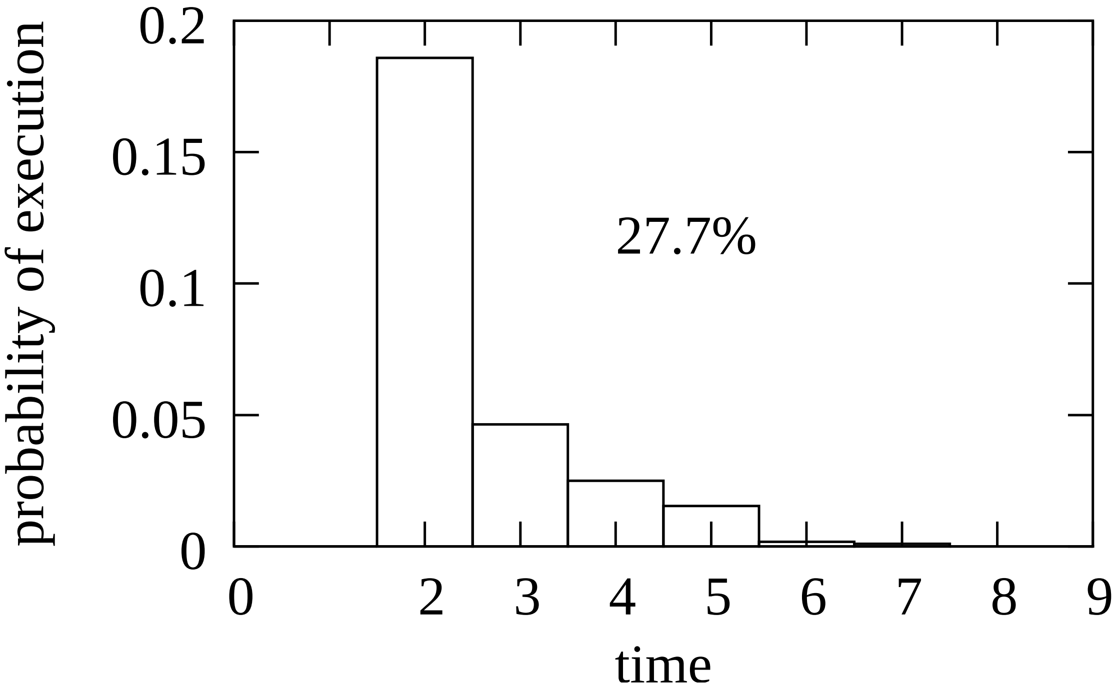
$$\tilde{V} = \begin{pmatrix} V \\ 1_{C(V)} \end{pmatrix} \tag{51}$$

The time process has to be adapted to test for the condition to be satisfied after each step.

$$\tilde{\Theta} \begin{pmatrix} V \\ x \end{pmatrix} = \begin{pmatrix} \Theta V \\ \Theta \begin{array}{l} \overset{C(V)}{\nearrow} \; 1 \\ \searrow \; x \end{array} \end{pmatrix} \tag{52}$$

Example

Probability of option execution. Due to the shorting time maturity the option becomes increasingly unlikely to be executed. The total probability of execution is 27.7%.



Parameter Fitting

Historical timeseries

$$\max_p \prod_i \prod_t (\Theta_p A_i)^M \delta(V(s_{t+1}))(V(s_t)) \tag{53}$$

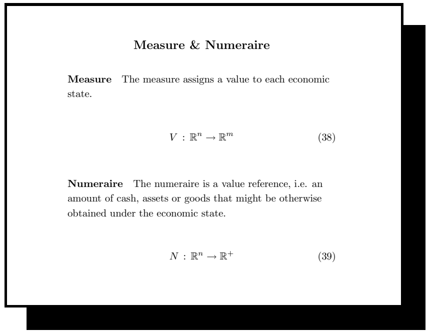
Implied parameters

$$\min_p \sum \left((\Theta_p A_i)^M N(s) - MarketValue_i\right)^2 \tag{54}$$

portfolio optimisation

$$\max_p (\Theta A_p)^M UN \tag{55}$$

A new financial calculus



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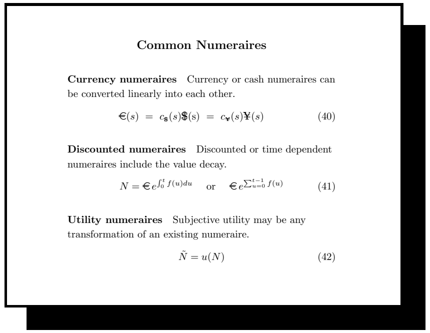

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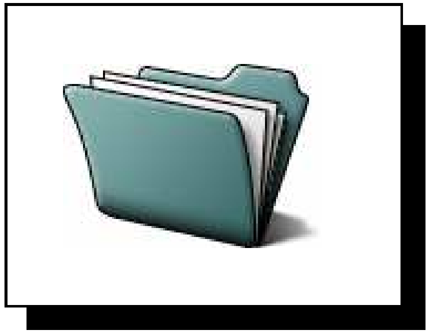

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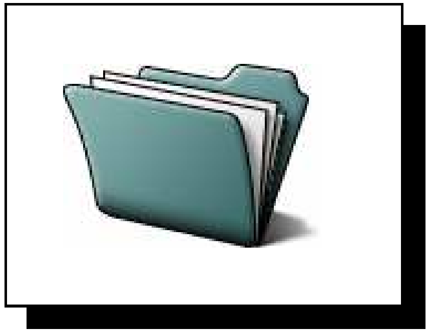

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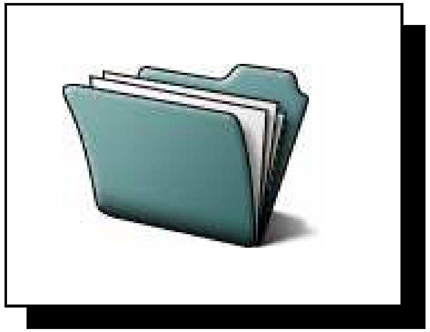

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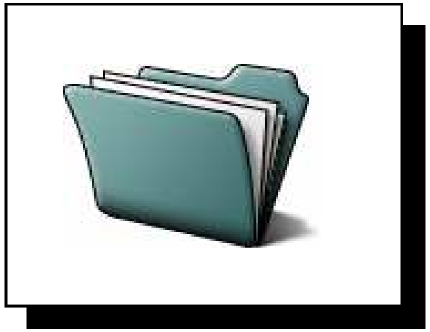


Parameter fitting

A new financial calculus

Introduction

⊞ Financial models

⊞ Implementation

Financial Model

A financial model is largely set up by three components: the process, the portfolio and a measure.

- The time **process** is a stochastic description of the path an observable variable can take.
- A **portfolio** describes a set of investments or the room for further trading activity.
- A **measure** is a computable property of the model. (E.g. expectations, distributions and parameters)

What is it?

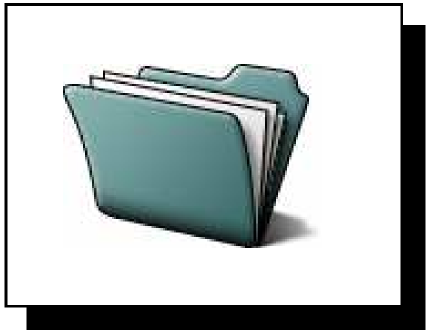
Notation

Suppose process and portfolio are each represented by an operator named Θ and A. The measure is a function V, such that

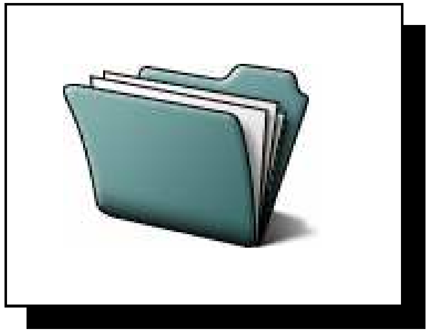
$$(\Theta\, A)^M\, V\,(s) \tag{7}$$

becomes a unique and easy to evaluate answer to the question: What is the measure V of portfolio A under process Θ over a horizon of length M, given a current economic state s.

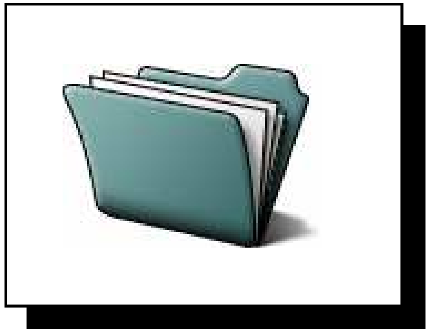
Notation



Process



Portfolio



Measure

A new financial calculus

Dadim

Example

Things to do

1. 大ML, an XML format for standardized model description
2. Monte Carlo engine that reads 大ML expressions
3. Increase and verify accuracy
4. Increase the speed of Dadim. (Adaptivity, Sparse grids, Operator sampling)
5. Stylesheet or XSLT to XHTML/MathML
6. Toolbox for standard models and standard products

Future

What is Dadim

Dadim is a purely discrete computer implementable function class that is isomorph in $\mathscr{L}^2$ to the function space $\mathbb{R}^{\mathbb{N}} \longrightarrow V$

$$\text{大}(V) \cong \mathscr{L}^2(\mathbb{R}^{\mathbb{N}} \longrightarrow V) \tag{56}$$

Dadim operators

Dadim can be defined algebarically as a 5-tuple consisting of a set of Dadims, an underlying vector space and three operators for evaluation, translation and dilatation.

$$\text{大\hspace{-0.55em}大}(V) = (\text{大\hspace{-0.55em}大}, V, \text{大}, \mathsf{T}, \mathsf{Z}) \tag{57}$$

The operators work on the domain as follows. T and Z are vectors of operators whereas each component acts in the according direction.

$$\begin{aligned} \text{大} &: \text{大\hspace{-0.55em}大} \to V \\ \mathsf{T} &: (\text{大\hspace{-0.55em}大} \to \text{大\hspace{-0.55em}大})^{\mathbb{N}} \\ \mathsf{Z} &: (\text{大\hspace{-0.55em}大} \to \text{大\hspace{-0.55em}大})^{\mathbb{N}} \end{aligned} \tag{58}$$

There are further restrictions on the behaviour of the operators, which are neglected in this talk.

Numerical interface

The domain and the three operators can implemented in Java an interface with three methods.

```
// Licensed under the Open Software License v1.1
public interface Dadim {

  public Object da();
  public void trans(int dir, int length);
  public Dadim zoom(int dir);
}
```

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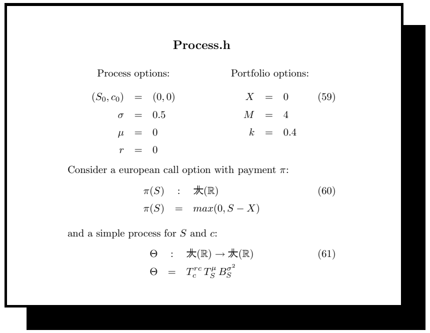
A new financial calculus



Process.h

Process options:

$$(S_0, c_0) = (0,0), \quad \sigma = 0.5, \quad \mu = 0, \quad r = 0$$

Portfolio options:

$$X = 0, \quad M = 4, \quad k = 0.4 \tag{59}$$

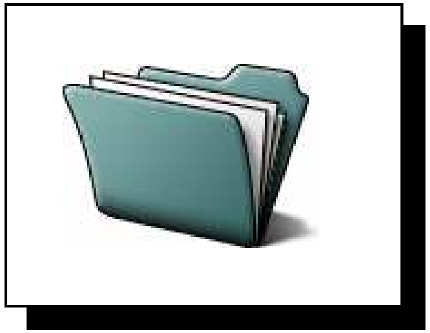
Consider a european call option with payment π:

$$\begin{aligned} \pi(S) &: 大(\mathbb{R}) \\ \pi(S) &= max(0, S - X) \end{aligned} \tag{60}$$

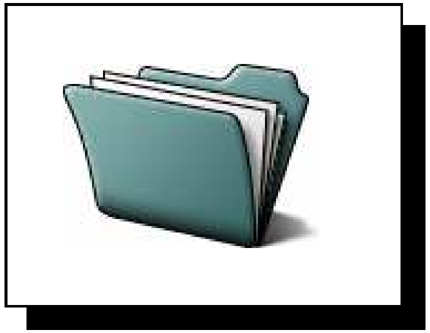
and a simple process for S and c:

$$\begin{aligned} \Theta &: 大(\mathbb{R}) \to 大(\mathbb{R}) \\ \Theta &= T_c^{rc} T_S^{\mu} B_S^{\sigma^2} \end{aligned} \tag{61}$$

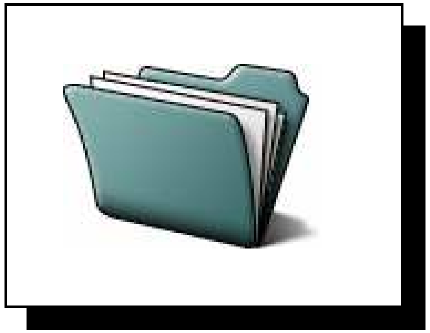
Process.h



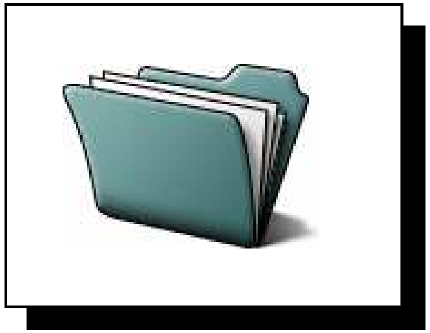
Price



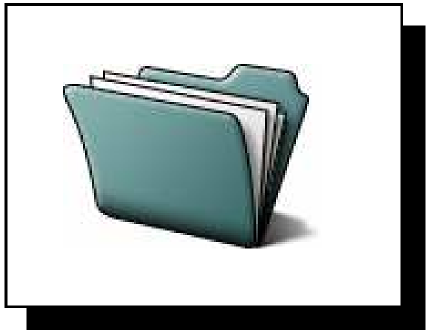
Distribution



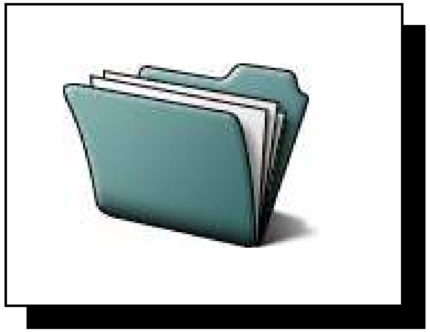
Variance



Delta-Hedge



Trading costs



Best hedge

Process.h

Process options: Portfolio options:

$$
\begin{aligned}
(S_0, c_0) &= (0, 0) & \qquad X &= 0 \\
\sigma &= 0.5 & \qquad M &= 4 \\
\mu &= 0 & \qquad k &= 0.4 \\
r &= 0 & &
\end{aligned}
\tag{60}
$$

Consider a european call option with payment π:

$$
\begin{aligned}
\pi(S) &: \text{大}(\mathbb{R}) \\
\pi(S) &= max(0, S - X)
\end{aligned}
\tag{61}
$$

and a simple process for S and c:

$$
\begin{aligned}
\Theta &: \text{大}(\mathbb{R}) \rightarrow \text{大}(\mathbb{R}) \\
\Theta &= T_c^{rc} T_S^{\mu} B_S^{\sigma^2}
\end{aligned}
\tag{62}
$$

A new financial calculus

Computing the expected value

The expectation value is computed as $\Theta^M(c + \pi(S))$:

V $\in$ 大$(\mathbb{R})$

V = c + $\pi(S)$
for t = 0 to M-1
    V = Θ V

V = Z^3 V
Result:
    大 V $\quad = \quad 0.389$

Distribution

$$f(c) = \Theta^M 1_{c>\pi(S)} \tag{63}$$

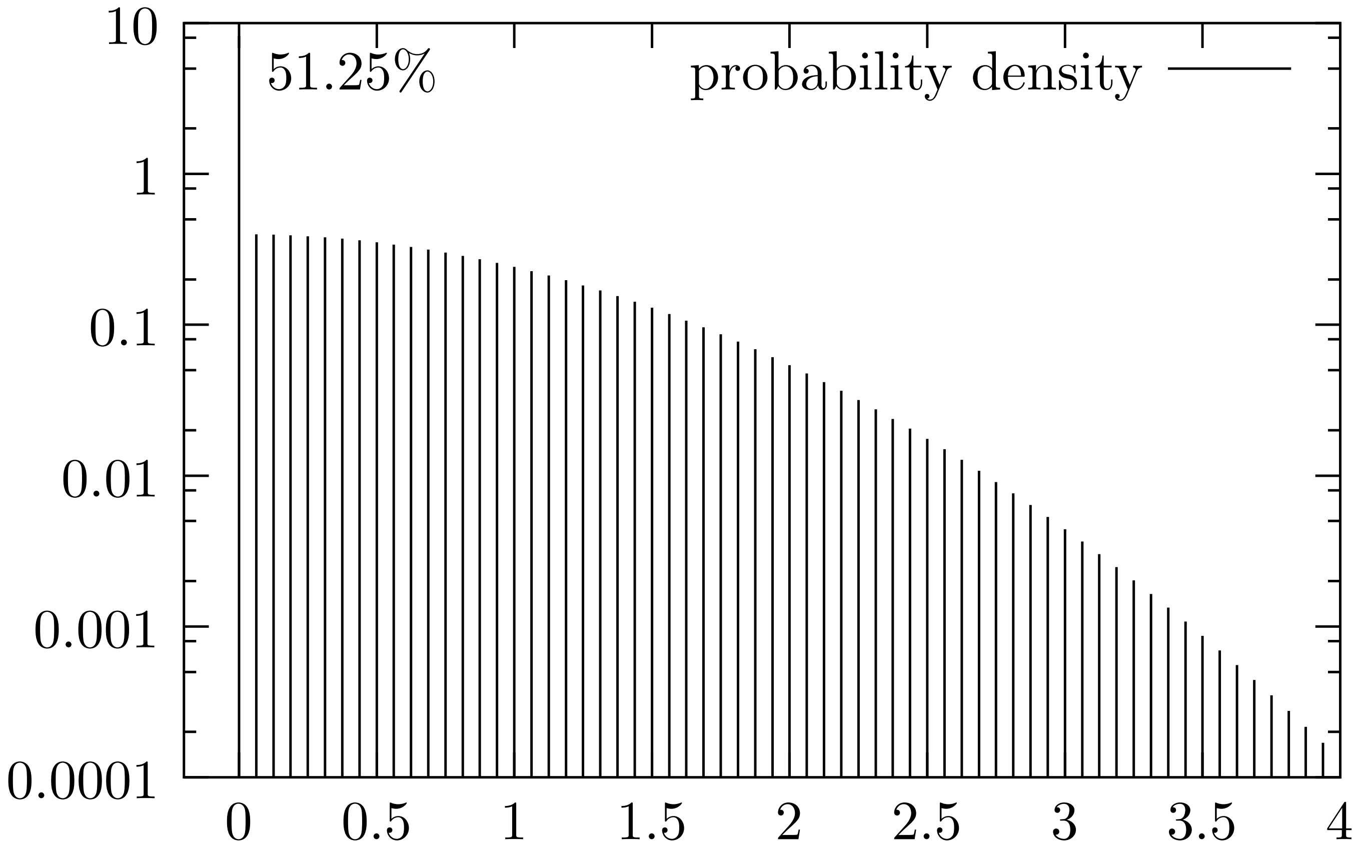
```
V = heaviside(c - π(S))
for t = 0 to M-1
     V = Θ V

V = d/dc V

V = Z_c^3 Z_S^3 V
Result:
     for x = 0 to ...
          大 V
          V = T_c V
```

Distribution

$$f(c) = \Theta^M 1_{c>\pi(S)} \tag{64}$$



Variance

Let's sell the option at the expected price 0.398 and compute the expected square deviation:

$$\Theta^M \Big(0.398 + c - \pi(S)\Big)^2 \tag{65}$$

```
var ∈ 大(ℝ)

var = pow(0.398 + c - π(S), 2)
for t = 0 to M-1
     var = Θ var

var = Z³ var
Result:
     大 var            =   0.34
     sqrt(大 var)      =   0.58
```

Delta-Hedge

The delta-hedge investment is defined by transaction of cach from account c into stocks in deposit h:

$$A_H V = (T_c^{-S} T_h)^{-\frac{d}{dS} V_1} \tag{66}$$

A two dimensional measure for expected value and variance:

$$(A_H \Theta)^M \begin{pmatrix} 0.398 + hS + c - \pi(S) \\ \big(0.398 + hS + c - \pi(S)\big)^2 \end{pmatrix} \tag{67}$$

Although, the delta-hedge is designed for continuous trading, there should be at least some reduction in the variance.

Delta-Hedge

`V` $\in$ 大$(\mathbb{R}^2)$

`a = 0.398 + c + h*S -` $\pi(S)$
`V = [a, pow(a,2)]`

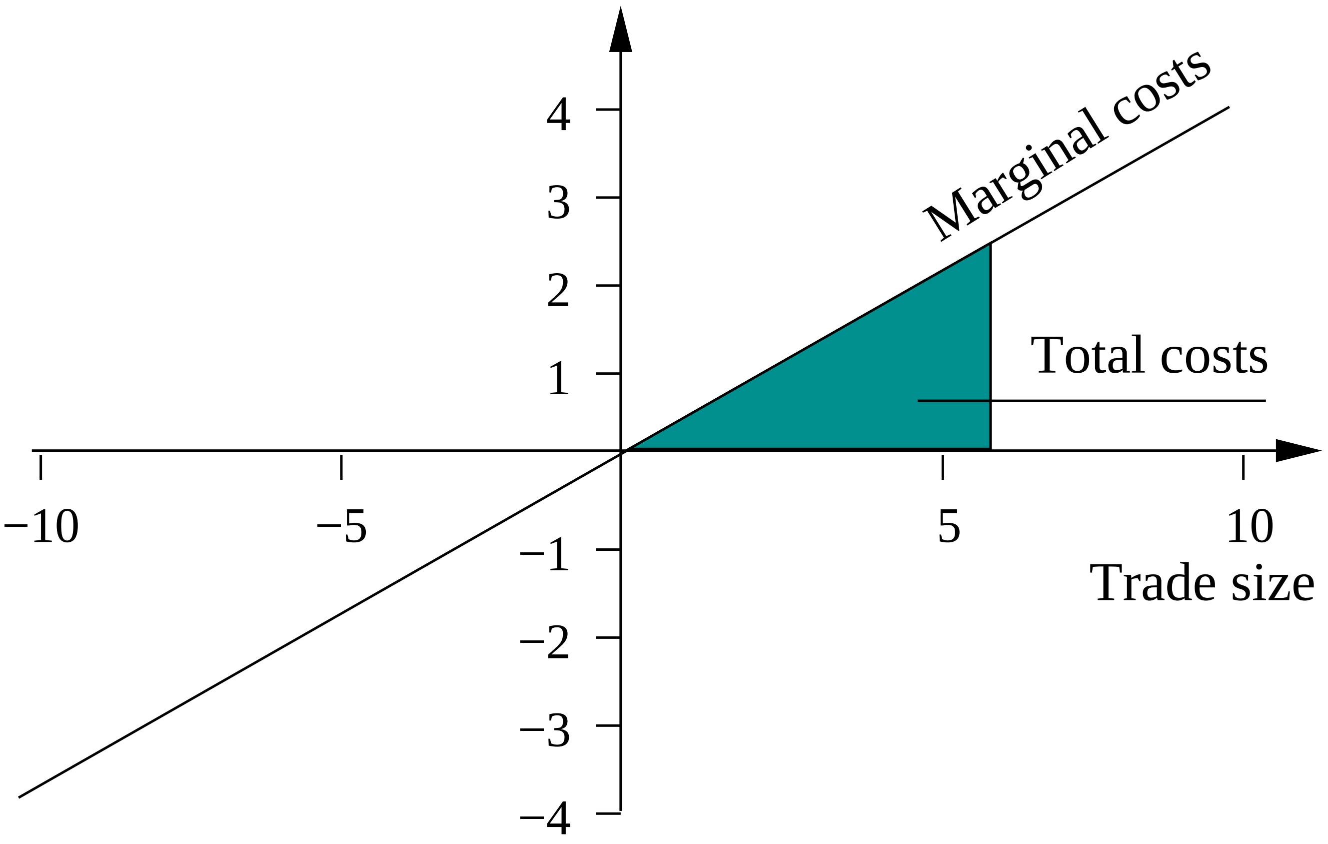
`for t = 0 to M-1`
  `V =` Θ `V`
  Δ `=` $\frac{d}{dS}$ `V[0]`
  `V =` $T_h^{-\Delta}\ T_c^{S\Delta}$ `V`

`V =` $\mathrm{Z}_h^2\ \mathrm{Z}_S^2$ `V`
`Result:`

大 `V[0]`	=	0.00
大 `V[1]`	=	0.03
`sqrt(`大 `V[1])`	=	0.17

Transaction costs

This model for the transaction costs assumes, that the price depends linearly on demand. The total costs are therefore proportional to the squared trade size.



Transaction costs

The hedge activity including transaction costs

$$A_H V = (T_c^{-S} T_h)^{-\frac{d}{dS}V_1} \underbrace{T_c^{-k\left(\frac{d}{dS}V_1\right)^2}}_{\text{pay costs}} V \tag{68}$$

Transaction costs

$\mathtt{V} \in 大(\mathbb{R}^2)$

```
a = 0.398 + c + h*S - π(S)
V = [a, pow(a,2)]

for t = 0 to M-1
      V = Θ V
```

$\Delta = \frac{d}{dS}\ \mathtt{V[0]}$

$\mathtt{V} = T_h^{-\Delta}\ T_c^{S\Delta - k\Delta^2}\ \mathtt{V}$

$\mathtt{V} = \mathtt{Z}_h^2\ \mathtt{Z}_S^2\ \mathtt{V}$

Result:

大 V[0]	=	−0.27
大 V[1]	=	0.12
sqrt(大 V[1])	=	0.34

A new financial calculus

Best hedge

a = 0.398 + c + h*S - $\pi(S)$

V = [a, pow(a,2)]

for t = 0 to M-1

    V = Θ V

    Δ = h* - h

    V = $T_h^{-\Delta-k\Delta^2}\ T_c^{S\Delta}$ V

    opt = solve($\frac{d}{dh^*}$ V[1] = 0, h*)

    V = $T_{h^*}^{\mathtt{opt}}$ V

V = $Z^2_{h^*}\ Z^2_S\ Z_h$ V

Result:

$$\text{大 V[0]} = -0.04$$
$$\text{大 V[1]} = 0.05$$
$$\text{sqrt(大 V[1])} = 0.23$$

Things to do

1. 大ML, an XML format for standardized model description
2. Monte Carlo engine that reads 大ML expressions
3. Increase and verify accuracy
4. Increase the speed of Dadim. (Adaptivity, Sparse grids, Operator sampling)
5. Stylesheet or XSLT to XHTML/MathML
6. Toolbox for standard models and standard products