Representing Portfolios

On the representation of trading strategies and financial portfolios

Stefan Dirnstorfer

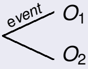
Technical University Munich

2004-Dec-13

Three Operators To Rule The Portfolio

All trading strategies and financial products can be described in terms of three operators.

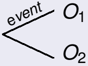
List of operators

	Initial values
Θ	An inactive time step
T_c	Transaction of one unit onto account c
	Decision between option O_1 and O_2
	Final valuation

Three Operators To Rule The Portfolio

All trading strategies and financial products can be described in terms of three operators.

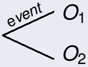
List of operators

	Initial values
Θ	An inactive time step
T_c	Transaction of one unit onto account c
	Decision between option O_1 and O_2
	Final valuation

Three Operators To Rule The Portfolio

All trading strategies and financial products can be described in terms of three operators.

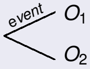
List of operators

	Initial values
Θ	An inactive time step
T_c	Transaction of one unit onto account c
	Decision between option O_1 and O_2
	Final valuation

Three Operators To Rule The Portfolio

All trading strategies and financial products can be described in terms of three operators.

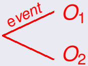
List of operators

	Initial values
Θ	An inactive time step
$T_c^{\mathbf{n}}$	Transaction of **n units** onto account *c*
	Decision between option O_1 and O_2
	Final valuation

Three Operators To Rule The Portfolio

All trading strategies and financial products can be described in terms of three operators.

List of operators

	Initial values
Θ	An inactive time step
T_c	Transaction of one unit onto account c
	Decision between option O_1 and O_2
	Final valuation

Portfolio Example: A Bond Deal

A multiperiod strategy can be written by a chronologically ordered operator sequence.

$$T_c^{-90} \ominus T_c^{100}$$

Read:

1. Withdraw 90 from account *c*
2. Wait one period
3. receive 100

Mathematica Code:

```
Function[V_, T[c, -90, Theta[ T[c, 100, V ]]]]
```

Portfolio Example: A Bond Deal

A multiperiod strategy can be written by a chronologically ordered operator sequence.

$$T_c^{-90} \ominus T_c^{100}$$

Read:

1. Withdraw 90 from account *c*
2. Wait one period
3. receive 100

Mathematica Code:

```
Function[V_, T[c, -90, Theta[ T[c, 100, V ]]]]
```

Portfolio Example: A Bond Deal

A multiperiod strategy can be written by a chronologically ordered operator sequence.

$$T_c^{-90} \ominus T_c^{100}$$

Read:

1. Withdraw 90 from account c
2. Wait one period
3. receive 100

Mathematica Code:

```
Function[V_, T[c, -90, Theta[ T[c, 100, V ]]]]
```

Portfolio Example: A Bond Deal

A multiperiod strategy can be written by a chronologically ordered operator sequence.

$$T_c^{-90} \ominus T_c^{100}$$

Read:

1. Withdraw 90 from account *c*
2. Wait one period
3. receive 100

Mathematica Code:

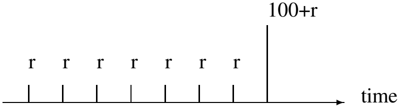
```
Function[V_, T[c, -90, Theta[ T[c, 100, V ]]]]
```

Portfolio Example: A Coupon Bond

Repeated actions can be denoted by the operator power.

$$\left(\Theta T_c^r\right)^M \; T_c^{100}$$

The operator term reads chronologically from left to right. The term in parenthesis is repeated M times.



Mathematica code

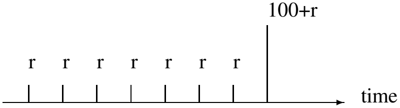
```
Function[V_,
Nest[ Function[f_, Theta[T[c, r, f] ]],  T[c, 100, V], M]]
```

Portfolio Example: A Coupon Bond

Repeated actions can be denoted by the operator power.

$$\left(\ominus T_c^r\right)^M \; T_c^{100}$$

The operator term reads chronologically from left to right. The term in parenthesis is repeated M times.



Mathematica code

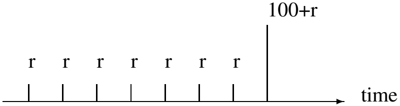
```
Function[V_,
Nest[ Function[f_, Theta[T[c, r, f] ]],  T[c, 100, V], M]]
```

Portfolio Example: A Coupon Bond

Repeated actions can be denoted by the operator power.

$$\left(\ominus T_c^r\right)^M \; T_c^{100}$$

The operator term reads chronologically from left to right. The term in parenthesis is repeated M times.



Mathematica code

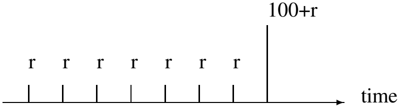
```
Function[V_,
Nest[ Function[f_, Theta[T[c, r, f] ]],  T[c, 100, V], M]]
```

Portfolio Example: A Coupon Bond

Repeated actions can be denoted by the operator power.

$$\left(\ominus T_c^r \right)^M \; T_c^{100}$$

The operator term reads chronologically from left to right. The term in parenthesis is repeated M times.



Mathematica code

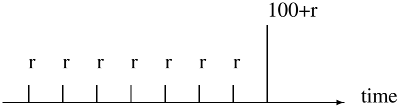
```
Function[V_,
Nest[ Function[f_, Theta[T[c, r, f] ]],  T[c, 100, V], M]]
```

Portfolio Example: A Coupon Bond

Repeated actions can be denoted by the operator power.

$$\left(\Theta T_c^r \right)^M \; T_c^{100}$$

The operator term reads chronologically from left to right. The term in parenthesis is repeated *M* times.



Mathematica code

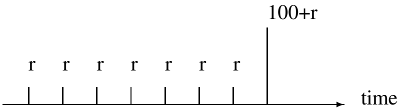
```
Function[V_,
Nest[ Function[f_, Theta[T[c, r, f] ]],  T[c, 100, V], M]]
```

Portfolio Example: A Coupon Bond

Repeated actions can be denoted by the operator power.

$$\left(\Theta T_c^r\right)^M \; T_c^{100}$$

The operator term reads chronologically from left to right. The term in parenthesis is repeated *M* times.

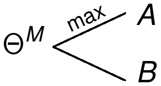


Mathematica code

```
Function[V_,
Nest[ Function[f_, Theta[T[c, r, f] ]],  T[c, 100, V], M]]
```

Portfolio Example: European Option

The European option offers the the choice between product A and B, after time to maturity M.



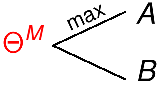
$$:= \Theta^M \max(A, B)$$

Mathematica code

```
Function[V_,
Nest[Theta,
 IfThen[ A[V]>B[V], A[V], B[V]],
M]]
```

Portfolio Example: European Option

The European option offers the the choice between product A and B, after time to maturity M.



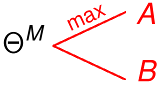
$$:= \Theta^M \max(A, B)$$

Mathematica code

```
Function[V_,
Nest[Theta,
  IfThen[ A[V]>B[V], A[V], B[V]],
M]]
```

Portfolio Example: European Option

The European option offers the the choice between product *A* and *B*, after time to maturity *M*.



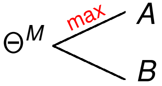
$$:= \Theta^{M}\,\max(A, B)$$

Mathematica code

```
Function[V_,
Nest[Theta,
  IfThen[ A[V]>B[V], A[V], B[V]],
M]]
```

Portfolio Example: European Option

The European option offers the the choice between product A and B, after time to maturity M.



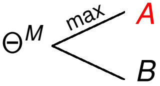
$$:= \Theta^M \max(A, B)$$

Mathematica code

```
Function[V_,
Nest[Theta,
  IfThen[ A[V]>B[V], A[V], B[V]],
M]]
```

Portfolio Example: European Option

The European option offers the the choice between product A and B, after time to maturity M.



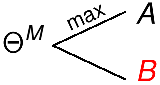
$$:= \Theta^M \max(A, B)$$

Mathematica code

```
Function[V_,
Nest[Theta,
  IfThen[ A[V]>B[V], A[V], B[V]],
M]]
```

Portfolio Example: European Option

The European option offers the the choice between product A and B, after time to maturity M.



$$:= \Theta^M \max(A, B)$$

Mathematica code

```
Function[V_,
Nest[Theta,
  IfThen[ A[V]>B[V], A[V], B[V]],
M]]
```

Portfolio Example: American Option

With American option there is a perpetual right to choose X.

$$\lim_{\Delta t \to 0} \left(\Theta^{\Delta t} \overset{\max}{\begin{cases} X \\ Id \end{cases}} \right)^{\frac{M}{\Delta t}}$$

Mathematica code for $\Delta t = 1$

```
Function[V_,
Nest[
Function[f_,
  Theta[ IfThen[X[V]>f, X[V], f]]],
V,M]]
```

Portfolio Example: American Option

With American option there is a perpetual right to choose X.

$$\lim_{\Delta t \to 0} \left(\Theta^{\Delta t} \overset{\max}{<} \begin{matrix} X \\ Id \end{matrix} \right)^{\frac{M}{\Delta t}}$$

Mathematica code for $\Delta t = 1$

```
Function[V_,
Nest[
Function[f_,
  Theta[ IfThen[X[V]>f, X[V], f]]],
V,M]]
```

Portfolio Example: American Option

With American option there is a perpetual right to choose X.

$$\lim_{\Delta t \to 0} \left(\Theta^{\Delta t} \overset{\max}{<} \begin{matrix} X \\ Id \end{matrix} \right)^{\frac{M}{\Delta t}}$$

Mathematica code for $\Delta t = 1$

```
Function[V_,
Nest[
Function[f_,
  Theta[ IfThen[X[V]>f, X[V], f]]],
V,M]]
```

Portfolio Example: American Option

With American option there is a perpetual right to choose X.

$$\lim_{\Delta t \to 0} \left(\Theta^{\Delta t} \max\{X, Id\} \right)^{\frac{M}{\Delta t}}$$

Mathematica code for $\Delta t = 1$

```
Function[V_,
Nest[
Function[f_,
  Theta[ IfThen[X[V]>f, X[V], f]]],
V,M]]
```

Portfolio Example: American Option

With American option there is a perpetual right to choose X.

$$\lim_{\Delta t \to 0} \left(\ominus^{\Delta t} \overset{\max}{<} \begin{array}{l} X \\ Id \end{array} \right)^{\frac{M}{\Delta t}}$$

Mathematica code for $\Delta t = 1$

```
Function[V_,
Nest[
Function[f_,
  Theta[ IfThen[X[V]>f, X[V], f]]],
V,M]]
```

Transfer operator

The transfer or shift operator T transfers a deterministic amount onto variable x.

$$T_x^n f(x) := f(x+n)$$

This operation replaces every instance of x with $x+n$.

Mathematica code

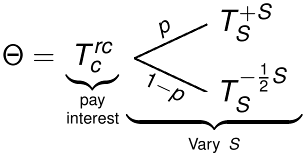
```
T[index_, power_, V_] := V /. index-> index+power
```

Process Operator

With Θ we can look one step into the future and evaluate the expectation of a function f under the future process state.

$$\Theta f(x) := \mathbb{E}\big(f(X_{t+1})\big|X_t = x\big)$$

Example:
Different process scenarios can be combined linearly.



Evaluation Sequence

We need to define initial process variables and determine the property of interest.

Initial values

Product or strategy

$$大 \overbrace{\Pi(\Theta)} V$$

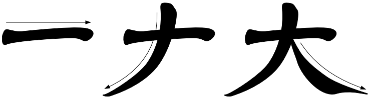
Final evaluation

Evaluation Operator

The chronological operator order is maintained by a new operator 大 that applies initial values from the left hand side.

$$\text{大}\, V = \underset{x=x_0}{\text{大}} V := V\big|_{x=x_0}$$

How to write:



Evaluation Of S

Starting with $S = 100$, what is S's expected value after one period?

$$\underset{S=100}{\text{大}} \ominus S = 200p + 50(1-p)$$

There exists a pseudo probability p, such that discounted S is a martingale.

$$\exists p: \underset{r=1/9}{\text{大}} \ominus S = S(1+r)$$

The result is found easily:

$$p = \frac{11}{27}$$

Evaluation Of S

Starting with $S = 100$, what is S's expected value after one period?

$$\underset{S=100}{\text{大}} \ominus S \; = \; 200p + 50(1-p)$$

There exists a pseudo probability p, such that discounted S is a martingale.

$$\exists p : \quad \underset{r=1/9}{\text{大}} \ominus S \; = \; S(1+r)$$

The result is found easily:

$$p = \frac{11}{27}$$

Evaluation Of S

Starting with $S = 100$, what is S's expected value after one period?

$$\underset{S=100}{\text{大}} \Theta\, S \;=\; 200p + 50(1 - p)$$

There exists a pseudo probability p, such that discounted S is a martingale.

$$\exists p : \quad \underset{r=1/9}{\text{大}} \Theta\, S \;=\; S(1 + r)$$

The result is found easily:

$$p = \frac{11}{27}$$

Pricing A Product

$$
\begin{aligned}
\underset{\substack{c=0\\ S=100}}{\text{大}}\ \underbrace{T_c^{-S}}_{\substack{\text{buy}\\ \text{stock}}}\ \underbrace{\Theta}_{\text{wait}}\ \underbrace{T_c^{+S}}_{\substack{\text{sell}\\ \text{stock}}}\ \underbrace{c}_{\text{profit?}} &= \text{大}\, T_c^{-S}\Theta\ c + S \\
&= \text{大}\, T_c^{-S}\ \frac{10(c+S)}{9} \\
&= \text{大}\, \frac{10c}{9} \quad = \quad 0
\end{aligned}
$$

$$
\underset{\substack{c=0\\ S=100}}{\text{大}}\ \underbrace{T_c^{-S}\,\Theta\, T_c^{+S}}_{\Pi(\Theta)}\ \underbrace{c^2}_{\text{risk?}} = 5432 \quad = \quad (73.7)^2
$$

Hedgeable price:	0€
Standard deviation:	73.7€

Pricing A Product

$$\underset{\substack{c=0\\ S=100}}{大}\ \underbrace{T_c^{-S}}_{\substack{\text{buy}\\ \text{stock}}}\ \underbrace{\Theta}_{\text{wait}}\ \underbrace{T_c^{+S}}_{\substack{\text{sell}\\ \text{stock}}}\ \underbrace{c}_{\text{profit?}} \;=\; 大\, T_c^{-S}\Theta\, c+S \;=\; 大\, T_c^{-S}\,\frac{10(c+S)}{9} \;=\; 大\,\frac{10c}{9} \;=\; 0$$

$$\underset{\substack{c=0\\ S=100}}{大}\ \underbrace{T_c^{-S}\Theta\, T_c^{+S}}_{\Pi(\Theta)}\ \underbrace{c^2}_{\text{risk?}} \;=\; 5432 \;=\; (73.7)^2$$

Hedgeable price: 0€

Standard deviation: 73.7€

Pricing A Product

$$\underset{\substack{c=0\\S=100}}{大}\ \underbrace{T_c^{-S}}_{\text{buy stock}}\ \underbrace{\ominus}_{\text{wait}}\ \underbrace{T_c^{+S}}_{\text{sell stock}}\ \underbrace{c}_{\text{profit?}} = 大\, T_c^{-S} \ominus c + S = 大\, T_c^{-S} \frac{10(c+S)}{9} = 大 \frac{10c}{9} = 0$$

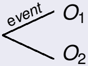
$$\underset{\substack{c=0\\S=100}}{大}\ \underbrace{T_c^{-S} \ominus T_c^{+S}}_{\Pi(\ominus)}\ \underbrace{c^2}_{\text{risk?}} = 5432 = (73.7)^2$$

Hedgeable price: 0€

Standard deviation: 73.7€

Five Operators To Rule Them All

List of operators

大	Initial values
Θ	An inactive time step
T_c	Transaction onto account c
	Decision between option O_1 and O_2
V	Final valuation