Theta−calculus for finance

Basic Operators

■ Shift operator

```
T[index_, power_, op_] :=
  ((op /. index → X00[t00]) /. DSolve[{X00'[t00] == (power /. index → X00[t00]),
        X00[0] == index}, X00[t00], t00][[1]][[1]]) /. t00 → 1
```

```
T[x, a, f[x]]
```

$f[a+x]$

```
T[x, r*x, f[x]]
```

$f[e^r x]$

■ Blur operator

```
B[index_, op_] :=
 Integrate[1/Sqrt[2*Pi]*Exp[-tt01^2/2]*(op /. index → index + tt01),
  {tt01, -Infinity, Infinity}]
```

```
B[index_, vola_, op_] :=
 Simplify[T[index, -X2*vola, B[X2, T[index, X2*vola, op]]],
  {TimeConstraint → 1}]
```

```
B[c, sigma, V[c]]
```

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{tt01^2}{2}} V[c + sigma\, tt01]}{\sqrt{2\pi}} \, dtt01$$

```
B[x, sigma, B[x, sigma, a + b*x + c*x^2]]
```

$a + b x + c (2\, sigma^2 + x^2)$

```
B[x, sigma*x, V[x]]
```

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{tt01^2}{2}} V[e^{sigma\, tt01} x]}{\sqrt{2\pi}} \, dtt01$$

■ Binomial blur

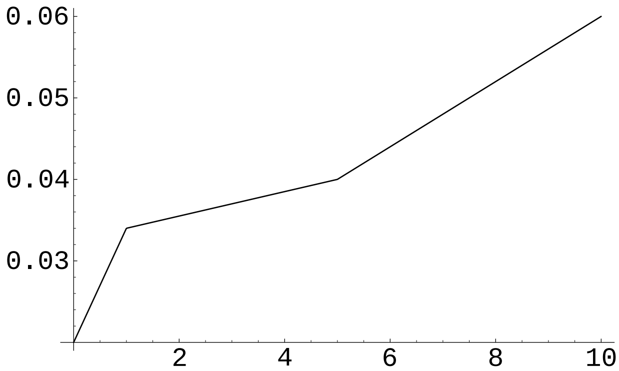
```
B2[index_, vola_, op_] :=
 Expand[(T[index, vola, op] + T[index, -vola, op]) / 2]

B2[x, 1, a + b * x + c * x^2]
```

a + c + b x + c x^2

Financial operators

■ Interest rate curves

```
Y0 := {{0, 0.02}, {1, 0.034}, {5, 0.04}, {10, 0.06}}

List2Curve[list_] := Function[M, If[M ≤ X2,
    ((X2 - M) * Y1 + (M - X1) * Y2) / (X2 - X1)
    , U1]] /.
  {X1 → list[[1]][[1]],
   X2 → list[[2]][[1]], Y1 → list[[1]][[2]], Y2 → list[[2]][[2]],
   U1 -> If[Length[list] > 2, List2Curve[Rest[list]][M], list[[2]][[2]]]}

Z0 := List2Curve[Y0];

Plot[Z0[x], {x, 0, 10}];
```



■ Initial values

```
Z1[M_] := M * (Z[M] + r1 * 1 / 100 + r2 * 1 / 100 * (M - 2));

X0 := {if → If, Z → Z0, t → 0, c → 0, r1 → 0, r2 → 0, eur → 1}

Da[V_] := V /. X0;
```

■ Theta operator

```
Theta[dt_, op_] :=
 T[eur, -eur * (Z1[t + dt] - Z1[t]),
  T[t, dt,
   op
  ]]
Theta[op_] := Theta[1, op]
```

```
Theta[5, t]
Da[%]
```

5 + t

5

```
Theta[10, eur]
Da[%]
N[-Log[%] / 10]
```

$e^{-\frac{r1}{10}-\frac{4 r2}{5}-\frac{r2 t}{5}+t Z[t]-10 Z[10+t]-t Z[10+t]}$ eur

0.548812

0.06

■ Interest rate extraction

```
forw[T1_, T2_] := Theta[T1, -Log[Theta[T2 - T1, eur] / eur] / (T2 - T1)]
```

```
Da[forw[3, 5]]
```

0.0445

```
-(Z0[3] * 3 - 5 * Z0[5]) / (5 - 3)
```

0.0445

```
Y[M_] := forw[0, M]
```

```
Da[Y[10]]
Da[Theta[3, Y[2]]]
```

0.06

0.0445

```
Da[Y[1 / 5]]
```

0.0228

■ Binomial tree Process

```
Theta2vola := 2 / 10
```

```
Theta2[V_] :=
 B2[r1, Theta2vola, Theta[V]]
Theta2[Dt_, V_] :=
 B2[r1, Theta2vola * Sqrt[Dt], Theta[Dt, V]]
```

■ Monte Carlo Process

```
Needs["Statistics`ContinuousDistributions`"]
```

```
randn := Random[NormalDistribution[0, 1]]
```

```
Table[randn, {i, 0, 9}]
```

```
{-0.707901, -0.867752, -2.46158, -0.961339,
 -0.25833, -1.22253, 1.56793, -1.10081, -0.22069, -0.187554}
```

```
Da[V_, n_] := Sum[Da[V /. N -> randn], {i, 1, n}] / n
```

```
Da[T[c, N, c], 1000]
Da[T[c, 1, T[c, c * N, c]], 1000]
```

```
0.00311872

1.62735
```

Products

Zero Bond

```
P := Theta[10, eur]
Da[P]
Da[T[r1, 1, P]]
Da[D[P, r1]]
Da[D[P, r2]]
Da[D[P, r1, r1]]
```

```
0.548812

0.496585

-0.0548812

-0.439049

0.00548812
```

Annuity

```
Op[V_] := Theta[T[c, a * eur, V]];
```

```
Op[c];
Da[%]
```

```
0.966572 a
```

```
Op[Op[c]]
Da[%]
```

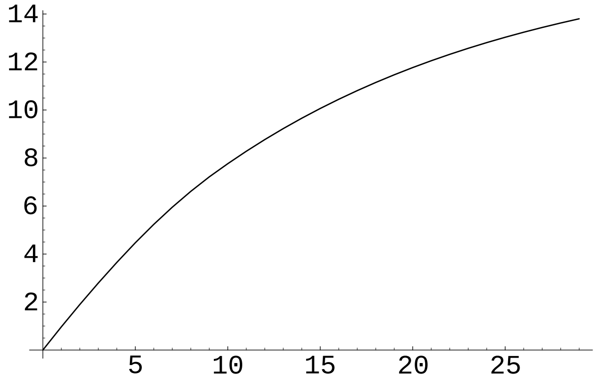
$$c + a\, e^{-\frac{r1}{100}+\frac{r2}{100}-\frac{r2\,t}{50}+t\,Z[t]-Z[1+t]-t\,Z[1+t]}\,\mathrm{eur} + a\, e^{-\frac{r1}{50}+\frac{r2}{50}-\frac{r2\,t}{50}-\frac{1}{50}\,r2\,(1+t)+t\,Z[t]-Z[1+t]-t\,Z[1+t]+(1+t)\,Z[1+t]-Z[2+t]-(1+t)\,Z[2+t]}\,\mathrm{eur}$$

```
1.89803 a
```

```
Nest[Op, c, 8];
Da[%]
```

```
6.61125 a
```

```
L[l_, V_] := If[l < 30, Append[L[l + 1, Op[V]], {l, Da[V]}], {}]
L2 := L[0, c] /. {eur → 1, a → 1}
ListPlot[L2, PlotJoined → True];
```



▪ Zero Bond Option

```
u := Evaluate[Theta[10, eur]];
u
Da[u]
```

$e^{-\frac{r1}{10}-\frac{4\,r2}{5}-\frac{r2\,t}{5}+t\,Z[t]-10\,Z[10+t]-t\,Z[10+t]}$ eur

0.548812

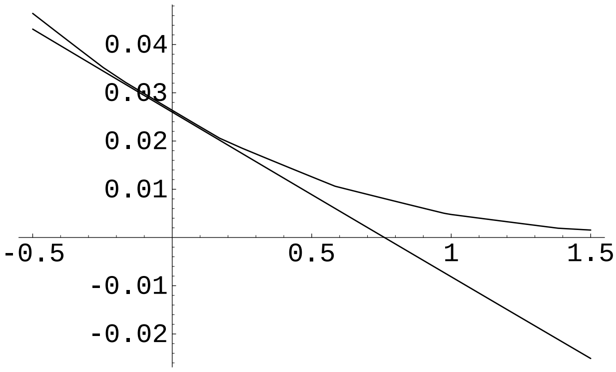
```
K := Da[u] - 1 / 100;
o := Function[{r1, r2}, eur * If[e^(1/10 (-r1-8 r2-2 r2 t+10 t Z[t]-100 Z[10+t]-10 t Z[10+t])) > K,
      e^(1/10 (-r1-8 r2-2 r2 t+10 t Z[t]-100 Z[10+t]-10 t Z[10+t])) - K, 0]];
Da[
 o[
  r1,
  r2]]
```

0.01

```
p := Nest[Theta2, o2[r1, r2], 20]
p /. o2 → o;
N[Da[%]]
```

0.0263722

```
p := Nest[Theta2, o2[r1, r2], 20]
pg := Evaluate[Da[T[r1, b, p] /. o2 → o]];
Plot[{pg,
  Da[T[r1, b, Theta[3.55 * eur]]] - Da[Theta[3.55 * eur]] + 0.026}, {b, -0.5, 1.5}]
```



- Graphics -

```
Da[D[p /. o2 → o, r1]]
Da[D[Theta[a * eur], r1]]
Solve[% == %%, a]
```

-0.0343247

-0.00966572 a

{{a → 3.55118}}

```
Da[D[p /. o2 → o, r2]]
```

-0.274597

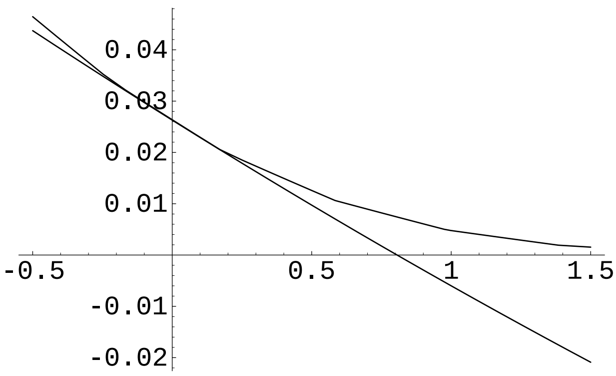
```
Da[D[Theta[a * eur + Theta[b * eur]], r1]]
Da[D[Theta[a * eur + Theta[b * eur]], r2]]
Solve[{%% == -0.034, % == -0.275}, {a, b}]
```

-0.00966572 a - 0.0186292 b

0.00966572 a

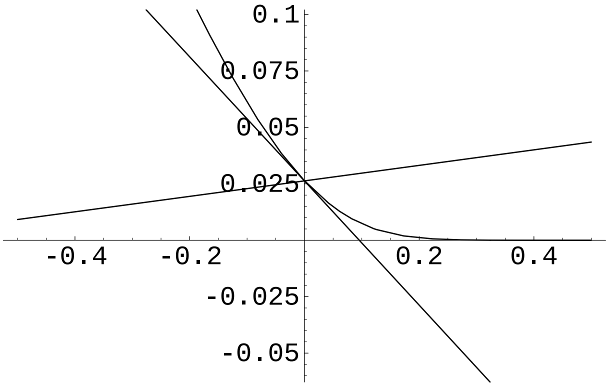
{{a → -28.4511, b → 16.5868}}

```
Plot[{Da[T[r1, b, Theta[-28.4511 * eur + Theta[16.5868 * eur]]]]
   - Da[Theta[-28.4511 * eur + Theta[16.5868 * eur]]] + 0.0263
  , pg}, {b, -0.5, 1.5}]
```



- Graphics -

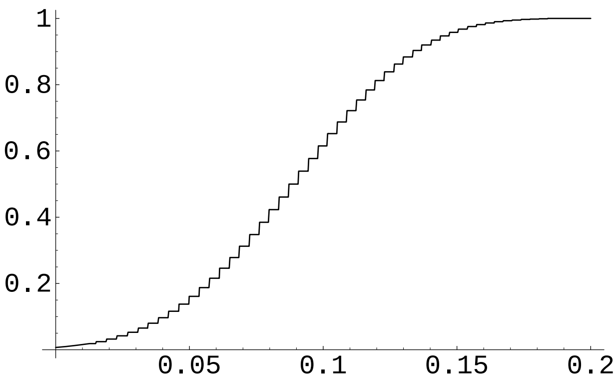
```
pg2 := Evaluate[Da[T[r2, b, p] /. o2 → o]];

Plot[{Da[T[r2, b, Theta[-28.4511 * eur + Theta[16.5868 * eur]]]]
    - Da[Theta[-28.4511 * eur + Theta[16.5868 * eur]]] + 0.0263,
   Da[T[r2, b, Theta[3.55 * eur]]]
    - Da[Theta[3.55 * eur]] + 0.0263,
   pg2
 }, {b, -0.5, 0.5}]
```




```
- Graphics -
```

■ Bond deal

```
Clear[q]; Clear[p]

q[V_] := T[c, -0.5 * eur, Nest[Theta2, T[c, 1 * eur, V], 10]];

p := q[f[eur]]

pvar := Evaluate[Da[q[If[c / eur > var, 0, 1]]]]

Plot[pvar
  , {var, 0.0, 0.2}];
```



■ index linked cash flow

```
Clear[ptheta]; Clear[p]

pdt := 5;
```

```
ptheta[V_] :=
 B2[index, Sqrt[1 / pdt] * 2 / 1000,
  T[index, index * 1 / 5 * Y[1 / pdt] / pdt,
   Theta2[1 / pdt, V]]]
p[V_] := Nest[ptheta, V, pdt] /. index → 1

payoff := eur * index;
```

■ Expected values

```
p[index];
Da[%]
```

```
1.00682
```

```
p[payoff];
Da[%]
```

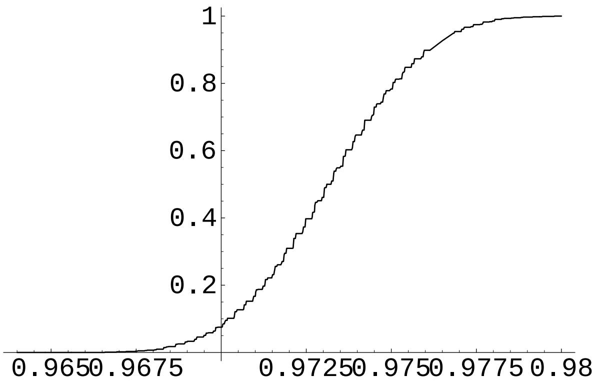
```
0.973167
```

```
Da[p[eur]]
```

```
0.966572
```

■ Distribtion

```
Clear[o]; Clear[o2]
o[V_] := If[V > var, 0, 1]

pvar := Evaluate[Da[p[o2[payoff]] /. o2 → o] /. eur → 1]

Plot[pvar
  , {var, 0.964, 0.98}];
```




```
m2 := Evaluate[Da[p[payoff^2]]]
m2
```

```
0.947059
```

```
m1 := Evaluate[Da[p[payoff]]]
m1
```

```
0.973167
```

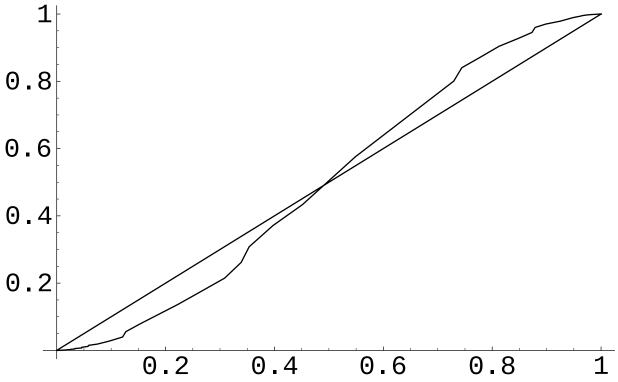
```
vola := Sqrt[Abs[m1^2 - m2]]
vola
```

```
0.00223369
```

■ QQPlot

```
pdist[x_] := .5 + .5 * Erf[(x - m1) / vola]
```

```
ParametricPlot[{{pvar, pdist[var]}, {pdist[var], pdist[var]}}, {var, 0.8, 1.0}]
```




```
- Graphics -
```

■ Sensitivity

```
Da[D[p[payoff], r1]]
```

```
-0.00778534
```

```
p1 := Da[p[payoff]]
p2 := Da[Theta[eur]]
p1
p2
```

```
0.973167
```

```
0.966572
```

```
Da[Theta[a eur]]
ps := Evaluate[Solve[% == p1, a]]
ps
```

```
0.966572 a
```

```
{{a → 1.00682}}
```

```
Da[D[Theta[(a /. ps[[1]]) eur], r1]]
```

```
-0.00973167
```

```
Solve[Da[D[Theta[b * eur], r1]] == -0.00778 eur, b]
```

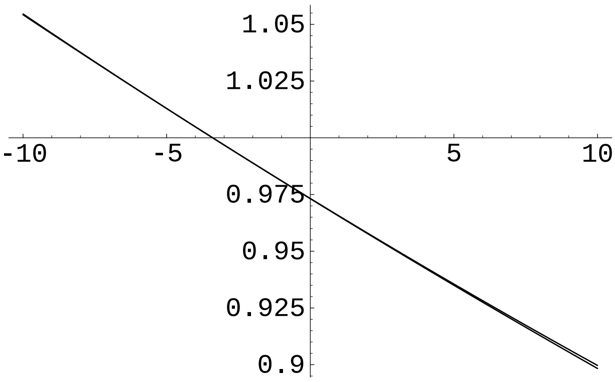
```
{{b → 0.804907 eur}}
```

```
modif := Da[-100 * D[p[payoff], r1] / p[payoff]]
modif
```

```
0.8
```

```
po := Evaluate[Da[T[r1, b, p[payoff]]] /. eur → 1]
bo := Evaluate[Da[T[r1, b, Theta[0.8 eur] + 0.2 eur]] /. eur → 1]
```

```
Plot[{po, bo}, {b, -10, 10}]
```

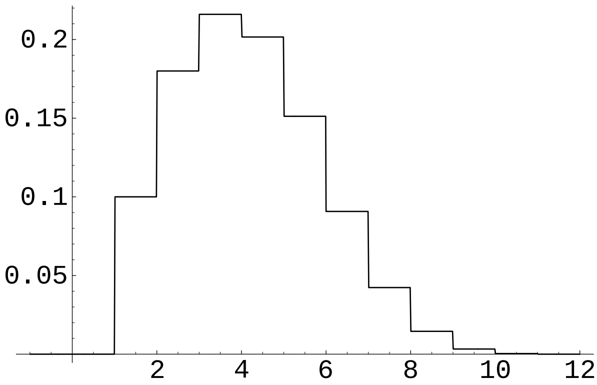



```
- Graphics -
```

Hazard rate model

```
Clear[ptheta]; Clear[p]

lambda := (0.1 * t) ^ (1)

ptheta[V_, pay_] := lambda * pay + (1 - lambda) * Theta[V]

p[V_] := Nest[Function[u, ptheta[u, V]], V, 12]

Da[p[t]]

3.66022

Da[p[eur]]
Da[D[p[eur], r1]]

0.865918

-0.0304832

ptheta2[V_] := B2[index, 2 / 1000, T[index, index * 0.2 * Y[1], ptheta[V, index * eur]]]

p2 := Evaluate[Nest[ptheta2, index * eur, 10] /. index → 1]

Da[p2]
Da[T[r1, 1, p2]] - %

0.899635

-0.0233727

stop := If[And[t < t0 , t > t0 - 1], 1, 0]
pstop := Nest[Function[u, ptheta[u, stop]], stop, 12]

General::spell1 :
 Possible spelling error: new symbol name "pstop" is similar to existing symbol "stop". More…
```

```
Evaluate[Da[pstop]];
Plot[%, {t0, -1, 12}]
```



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