On the representation of trading strategies and financial products

Stefan Dirnstorfer

Institute for Mathematical Finance
Technical University Munich

IWIF Melbourne, 2004

Abstract

This paper considers sequences of trading activities and suggests a mathematical representation of typical pattern in human investment decisions. All kinds of contracts and strategies can be captured in terms of quantitative implications by a vocabulary of three words: waiting, transacting and deciding. Each activity will be represented by an operator, that can be interpreted in an operator sequence as a chronologically ordered list of instructions. In an example setting we will investigate the behavior of a typical risk averse agent and derive hedging possibilities and the resulting economic impact.

0 Introduction

The field of quantitative finance has to deal with an increasing complexity and the rapid development of new contract types. The creation of a consistent and unified portfolio framework is complicated by the lack of a mathematical notation for human trading activities and financial products. Elaborate contracts and investment objectives are mostly specified in prose form and typically use specific terminology that is hard to interpret by an uninvolved. Such representations are difficult to evaluate mathematically and have to be translated into formulas and computer code individually. Those who feel inclined to pursuit greater generality are mostly struck by an inflation of parameters and mathematical concepts.

In order to meet the industry demand for a technical portfolio representation there has been development in the extension of existing programming dialects. Most notable results are *FiML* [3], based on the functional programing language Caml, and the XML-standard *Fpml* [1]. However, neither provides a mathematical framework for the derivation of theoretical properties. Their vocabulary is huge and gets still extended. Finally, there is a large distance from product representation to the evaluation procedure, which raises the fear of model ambiguity and inconsistency.

This document suggests an explicit and mathematically precise operator notation for financial portfolios and financial derivatives. The notation is based on the foundations of operator theory and introduces a vocabulary of three operators: waiting, transacting and deciding. Each of the possible activities is represented by an operator, which is written in a chronological list to express a sequence of trading activities. The notation thus provides explicit expressions for contract details and trading strategies including embedded options, minimum guarantees, event triggers and non-delta hedges. Furthermore the notation yields a rather explicit procedure to extract its statistical properties, which can be performed by a computer algebra system or by a numerical scheme that is directly derived from the operator sequence.

1 The economic State

The prevailing state of economy at time t is summarized by a vector $X(t)$, that contains all relevant and available knowledge. This includes observable market parameters and private book keeping variables. For the scope of this document we will focus on a number of parameters that will occur in the following examples. As relevant information $X(t)$ we consider the current time t, the stock price S, the short rate r, our wealth in cash c and the number of stocks h in our deposit.

$$X(t) = \big[t, S(t), r(t), c(t), h(t), \cdots\big] \in \mathbb{R}^n \tag{1}$$

Other parameters that might occur as economic state are forward rates, exchange rates, stochastic volatilities, moving averages and default probabilities, if not constant. In short, every variable that varies in time must be represented as a component of X.

2 Valuation function

A valuation function V is an interpretation of state X. It determines the kind of information we want to extract from a portfolio. The next section will show how to compute the expected value of such a function, after some random events occurred. V might evaluate theoretical contract prices, risk measures or probabilities for certain events to happen after an initial state.

$$V : \mathbb{R}^n \to \mathbb{R}^m \tag{2}$$

We will briefly discuss the three most common instances of the valuation function. The first example V_c evaluates the amount on the cash account c. It is applicable whenever we are interested in expected cash profits.

$$V_c\big(X(t)\big) = c(t) \tag{3}$$

Additionally, we can take our stock deposit into the balance sheet. The function V_a returns the nominal worth of the stock deposit plus the cash account.

$$V_a\big(X(t)\big) = S(t)h(t) + c(t) \tag{4}$$

Finally, we consider a probability measure, that is an indicator function on the cash account greater than x. If we compute the expected value of the indicator function V_p we can determine the probability distribution of our total cash profit.

$$V_p\big(X(t)\big) = 1_{c(t)>\mathrm{x}} \tag{5}$$

All risk measures like value at risk and expected short fall can be derived from the full knowledge of the probability distribution.

3 Processes

The time process operator Θ is a function of a function and determines the stochastic model for the state variables. With $\Theta^{\Delta t}$ we can look one step Δt into the future and evaluate the expectation of our function V under the future process state. Whenever the operator occurs in a sequence of event operators it refers to a time step Δt with no activity.

$$\Theta : (\mathbb{R}^n \to \mathbb{R}^m) \to (\mathbb{R}^n \to \mathbb{R}^m) \tag{6}$$

The process operator $\Theta^{\Delta t}$ is defined as the expected value of the argument function V applied to tomorrow's state $X(t+\Delta t)$ given a current state $\mathrm{x} = X(t)$. The operator always corresponds to a Markovian process, or one that can be turned into Markovian form.

$$\Theta^{\Delta t} V(\mathrm{x}) := \mathbb{E}\left[V\big(X(t+\Delta t)\big) \Big| X(t) = \mathrm{x} \right] \tag{7}$$

Another version of the same operator is given by the probability density $p_{\Delta t}(\mathrm{x}, \mathrm{y})$ for the state to travel from x to y within one time step. Explicit formulas for this density are derived below for the most common processes.

$$\Theta^{\Delta t} V(\mathrm{x}) = \int_{\mathbb{R}} p_{\Delta t}(\mathrm{x}, \mathrm{y}) V(\mathrm{y}) d\mathrm{y} \tag{8}$$

Lemma: Multiple applications of the process $\Theta^{\Delta t}$ evaluate the same expected value as (7), but with the new time horizon in the exponent.

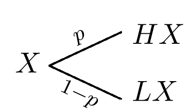
$$\left(\Theta^{\Delta t}\right)^n = \Theta^{n\Delta t} \tag{9}$$

Proof: The correctness of the operator power rule (9) is verified through repeated application of Θ according to its definition (7).

$$\begin{aligned}
(\Theta^{\Delta t})^n V(\mathrm{x}) &= (\Theta^{\Delta t})^{n-1}\, \mathbb{E}\left[V(X(t+\Delta t)) \big| X(t) = \mathrm{x} \right] \qquad (10) \\
&= (\Theta^{\Delta t})^{n-2}\, \mathbb{E}\left[\mathbb{E}\left[V(X(t+2\Delta t)) \big|_{X(t+\Delta t) = X(t+\Delta t)} \right] \big| X(t) = \mathrm{x} \right] \\
&= (\Theta^{\Delta t})^{n-2}\, \mathbb{E}\left[V(X(t+2\Delta t)) \big| X(t) = \mathrm{x} \right] \\
&\ \vdots \\
&= \mathbb{E}\left[V(X(t+n\Delta t)) \big| X(t) = \mathrm{x} \right] \qquad = \Theta^{n\Delta t} V(\mathrm{x})
\end{aligned}$$

3.1 Discrete process

A simple model but yet a good approximation to reality is the discrete state model. Suppose a random variable X reaches the high state HX with probability p and drops to the low state LX otherwise.



(11)

The expected value of a measuring function V is the linear combination of both scenarios. That is just how the expected value was defined in statistics.

$$\Theta V(X) := pV(HX) + (1-p)V(LX) \tag{12}$$

Of course we can build multistep trees through multiple applications of Θ. The example below shows the first two steps.

$$\begin{aligned}\Theta^2 V(X) &= \Theta\left[pV(HX) + (1-p)V(LX)\right] \\ &= p\Theta V(HX) + (1-p)\Theta V(LX) \\ &= p^2 V(HHX) + p(1-p)V(LHX) + \\ &\quad\; p(1-p)V(HLX) + (1-p)^2 V(LLX)\end{aligned} \tag{13}$$

The result simplifies in recombining trees where the operators L and H commute $LH = HL$.

3.2 Lévy processes

The Lévy process operator is defined by a convolution with the Lévy density, that is determined by its characteristic function ϕ. The convolution kernel is extracted by a Fourier transformation of the Euler constant to the power of ϕ.

$$L^{\phi} V(\mathrm{x}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{e^{\phi(y)}} V(\mathrm{x}-y)dy \tag{14}$$

The function ϕ is the logarithm of the Fourier transformed density and is uniquely determined by the Lévy triplet $[\mu, \sigma, \nu]$. Whereas μ is the drift, σ the volatility of a Brownian motion and ν is the Lévy measure reflecting the intensity of jumps of different sizes.

$$\phi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} e^{iu\mathrm{x}} - 1 - iu\mathrm{x}\, 1_{|\mathrm{x}|<1} \nu(d\mathrm{x}) \tag{15}$$

All processes that that can be homogeneously split into arbitrary time steps Δt, with $p_{\Delta t}(\mathrm{x}, \mathrm{y}) = p_{\Delta t}(\mathrm{x} - \mathrm{y})$, are Lévy processes. The most famous is the Brownian motion with $\nu = 0$.

Lemma: The super position ϕ to the operator L satisfies the algebraic power rule. This allows intuitive calculations with operator L and its exponent ϕ.

$$\left(L^{\phi}\right)^n = L^{n\phi} \tag{16}$$

Proof: The proof utilizes the fact that the convolution formula (14) turns into pointwise multiplication in Fourier space.

$$\begin{aligned}\left(L^{\phi}\right)^n V &= \widehat{e^{\phi}} * \cdots * \widehat{e^{\phi}} * V \\ &= \widehat{e^{n\phi}} * V \qquad = L^{n\phi} V\end{aligned} \tag{17}$$

Formula (16) also defines a default root for non integer exponents n. It is an exclusive property of Lévy processes that the convolution kernels generated by fractional powers are in fact positive probability densities.

3.3 The Black&Scholes model

The Black&Scholes model is a theoretical framework for the dynamics of stock prices and is used for the valuation of stock options. Although options are the topic of the next section on financial contracts, we will briefly discuss how the Θ operator is applied in this context. The profit V that we can draw from a call or put option depends on the difference between stock price S and strike price K. In case of a call option we have the right to buy one share at price K. Our profit is consequently $S - K$. The profit is multiplied with the discount factor e^{-rt}.

$$V_{call}(S,t) = \max(S-K,0)e^{-rt} \tag{18}$$

In the notation of partial differential equations we write the change of the expected value $\Theta^M V$ in its classical form. The equation yields a unique solution for the expected option value at expiration time.

$$\frac{d}{dM}\Theta^M V = \frac{d}{dt}\Theta^M V + rS\frac{d}{dS}\Theta^M V + \frac{1}{2}\sigma^2 S^2 \frac{d^2}{dS^2}\Theta^M V \tag{19}$$

The solution of this equation is well known and can be written explicitly by a convolution with the Gaussian density. The operator Θ_{bs} solves the Black&Scholes formula for the unit time step.

$$\Theta_{bs}\, V(S,t) = \int_{\mathbb{R}} \frac{e^{-\mathrm{x}'^2/2}}{\sqrt{2\pi}} V\left(Se^{r+\sigma \mathrm{x}' - \frac{1}{2}\sigma^2}, t+1\right) d\mathrm{x}' \tag{20}$$

The expectation of the option payoff at maturity time M is computed by an application of Θ with the appropriate power. The resulting function is supplied with the initial values for S and t.

$$\Theta_{bs}^M V_{call}(S_0, 0) \tag{21}$$

Note that the operators are placed in chronological order from left to right. First, you wait M time steps, and then you evaluate the payoff. Although the operators are written left to right, their evaluation direction is opposite. First compute the value function V and then apply Θ_{bs} M times.

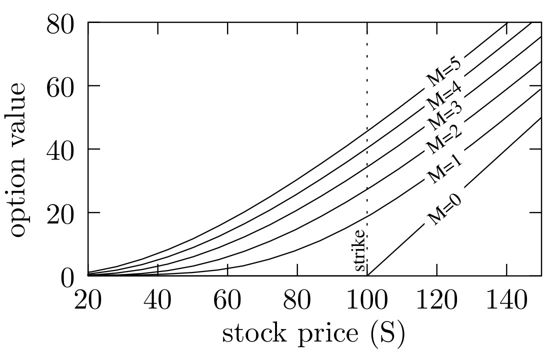


Figure 1: Call option under the Black&Scholes model. The process operator Θ_{bs} is applied M times to V_{call}

Plotted function:
$f_M(S) = {\Theta_{bs}}^M\, V_{call}(S,0)$
with
$\sigma = 40\%,\ r = 5\%,\ K = 100$

The Black&Scholes model describes a stock that increases its value in expectation with the same rate as an interest rate account. The great achievement of Black and Scholes was that options prices can be computed as the expected pay off under the adjusted drift although real stock prices are experienced to grow at a significantly higher rate. Only these option prices can be reproduced without risk in a continuous buying and selling strategy called delta hedge.

4 The Portfolio

The ultimate goal of this document and the presented operator notation is the ability to describe complex portfolios and contract conditions. We wanted to derive a mathematical term that fully represents our portfolio, including all outstanding transactions, all embedded options, the sensitivity to random events and the room for further trading activity.
A portfolio Π is a function of an operator. Given an operator Θ this results in the new portfolio operator $\Pi(\Theta)$. The effect of the transformation Π is simply an expansion of the considered state variables. While Θ only evaluates the expectation of observable market parameters without any interaction, the portfolio $\Pi(\Theta)$ expands the statistic measure to balance and accounting variables that correspond to the portfolio or the trading strategy.

$$\Pi : ((\mathbb{R}^n \to \mathbb{R}^m) \to (\mathbb{R}^n \to \mathbb{R}^m)) \to ((\mathbb{R}^n \to \mathbb{R}^m) \to (\mathbb{R}^n \to \mathbb{R}^m)) \tag{22}$$

A portfolio is specified by a vocabulary of three words: waiting, transacting and deciding. A time step without activity is denoted by Θ. For the transaction we will see the T operator. Decisions will be expressed by a specially designed option operator.

4.1 Transaction

The transaction operator T transfers one unit to the accounting variable specified by the operator index. Accordingly, T_{x_i} increases the i-the component of the state vector x by one. We can use the operator power to transfer multiple goods at once.

$$T_{\mathrm{x}_i} V(\mathrm{x}) := V(\mathrm{x} + \mathbf{e}_i) \tag{23}$$

Let our value V depend on the amount of units in the account x and some other state variables y. The new value after the delivery of a units is $T_{\mathrm{x}}^a V$. The number of transfered goods may vary with the states x and y.

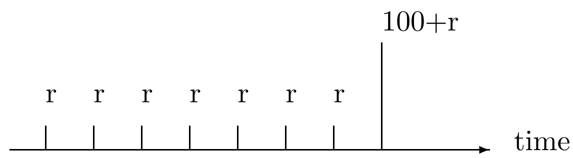
$$T_{\mathrm{x}}^{a(\mathrm{y})} V(\mathrm{x}, \mathrm{y}) = V(\mathrm{x} + a(\mathrm{y}), \mathrm{y}) \tag{24}$$

This operator replaces every occurrence of the index variable with the same variable plus the exponent.

Coupon bond The notation of a coupon bond highlights the use of the operator power to express repeated transactions. The operator term reads chronologically from left to right. First pay a regular coupon rate of size r until maturity M, then pay a redemption of 100.

$$\Pi_c(\Theta) = (\Theta T_c^r)^M T_c^{100} \tag{25}$$

The cash flow of a coupon bond is visualized in the plot below.



Swap A swap is an arrangement in which two parties repeatedly exchange assets at predefined conditions. In the case of currency swaps they exchange certain amounts in different currencies over a certain period. For interest rate swaps one pays a fixed size while the other pays a state dependent number. The swap investment Π_s simultaneously transacts r_1 in one direction and r_2 to opposite side over M periods.

$$\Pi_s(\Theta) = \left(\Theta\, T_c^{-r_1}\, T_c^{r_2}\right)^M \tag{26}$$

Swaps are usually agreed on without any initial premium. The expected profit from a swap is zero for both parties.

$$\Pi_s(\Theta)\, V_c = 0 \tag{27}$$

4.2 Option

An option is defined by the alternatives among which can be selected and by the entity that does select. Feasible choices are specified by the two portfolios Π_1 and Π_2. Depending on the choice, one of the optional portfolios determines the remaining portfolio after the option expired. The deciding entity is characterized by her choice condition C. If C is one the first option Π_1 is chosen. If C is zero, Π_2 is the selected portfolio.

$$\begin{array}{l} \overset{C}{\nearrow}\ \Pi_1 \\ \underset{1-C}{\searrow}\ \Pi_2 \end{array} := C\Pi_1 + (1-C)\Pi_2 \tag{28}$$

The choice condition C is an operator. When applied to a value function V it returns an indicator function on the states that lead to choice Π_1. In the most common cases the value function itself carries the information on which choice is preferred. If the option is long, i.e. the holder of the product herself can choose, we can use the choice condition $C_{\max}$ to yield the scenario with higher value.

$$C_{\max}V = \Pi_1 V > \Pi_2 V \tag{29}$$

In case of a short option, someone else can choose. Normally this leaves us with less valuable scenario $C_{\min}$.

$$C_{\min}V = \Pi_1 V < \Pi_2 V \tag{30}$$

Bond option Combining the option and the transaction operator we can specify our first interest rate derivative. The operator term below describes an option on a zero coupon bond. Again, the term reads chronologically from left to right. First, we wait a time m and then choose between two scenarios.

In the first scenario we first pay a strike price K and wait for the underlying maturity M until we receive a final redemption of 100. The second scenario is the identity operator and refers to no transaction.

$$\Theta^m \left\langle \begin{array}{l} \overset{C}{\nearrow}\; T_c^{-K}\Theta^M T_c^{100} \\ \underset{1-C}{\searrow}\; \cdot \end{array}\right. \tag{31}$$

Asian option The Asian option, as denoted below, is easily read from left to right. First we repeat n times a time step that waits one period and then adds the current stock price S to an accounting variable a. Finally we can choose to receive the difference of the average stock price a/n and the strike price K as a cash payment.

$$\left(\Theta T_a^S\right)^n \left\langle \begin{array}{l} \overset{C}{\nearrow}\; T_c^{a/n-K} \\ \underset{1-C}{\searrow}\; \cdot \end{array}\right. \tag{32}$$

4.2.1 Random events

A special case of the option operator occurs if the choice condition is unobservable. Economy knows a lot of sudden and unforeseeable events. Companies can default and outstanding payments become void or unexpected damages have to be compensated. In this case we can use the same operation as for the option. The only difference is that we use a choice probability λ instead of the deterministic condition.

$$\left\langle \begin{array}{l} \overset{\lambda}{\nearrow}\; \Pi_1 \\ \underset{1-\lambda}{\searrow}\; \Pi_2 \end{array}\right. := \lambda\Pi_1 + (1-\lambda)\Pi_2 \tag{33}$$

Subdivision We assume an event that occurs with probability λ per period. If we want to decide whether the event occurred after a non unit time step of Δt we use the probability $\lambda^{\Delta t}$ instead of λ. For $\Delta t = 1/2$ we can easily verify that the event operator with $\sqrt{\lambda}$ is the square root of the event operator with probability λ.

$$\left(\left\langle \begin{array}{l} \overset{\sqrt{\lambda}}{\nearrow}\; \cdot \\ \underset{1-\sqrt{\lambda}}{\searrow}\; X \end{array}\right. \right)^2 V = \left\langle \begin{array}{l} \overset{\sqrt{\lambda}}{\nearrow}\; \left\langle \begin{array}{l} \overset{\sqrt{\lambda}}{\nearrow}\; V \\ \underset{1-\sqrt{\lambda}}{\searrow}\; X \end{array}\right. \\ \underset{1-\sqrt{\lambda}}{\searrow}\; X \end{array}\right. = \lambda V + (1-\lambda)X \tag{34}$$

4.2.2 Time continuous option

The time continuous option, also referred to as American option, is an option with the continuous right of exercise. Over a certain time span M the holder of the option has the continuous right to leave the product path and branch into the optional product Π_X. The option requires a time process that can be infinitely subdivided and an option that is exercisable after each infinitely small time step. The American option is defined by the limit of the number of time steps to infinity and the length of each step Δt to zero.

$$\lim_{\Delta t \to 0} \left(\Theta^{\Delta t} \begin{array}{l} \overset{C}{\nearrow} \; \Pi_X \\ \underset{1-C}{\searrow} \; . \end{array} \right)^{M/\Delta t} \tag{35}$$

As an alternative to the limit function we can write the same term with the differential form dt replacing the discrete time step Δt.

$$\left(\Theta^{dt} \begin{array}{l} \overset{C}{\nearrow} \; \Pi_X \\ \underset{1-C}{\searrow} \; . \end{array} \right)^{M/dt} \tag{36}$$

American stock option The American stock option consists of a cash transaction for the choice Π_X, which when executed ends the investment. First, the amount $S - K$ is transfered. Then the value function V ends the operator term. The function can be considered as a constant operator that evaluates V regardless of what function it is applied to.

$$\Pi_X = T_c^{S-K} V \tag{37}$$

4.2.3 Space continuous option

Space continuous options require continuous values to fully describe the selected action. A portfolio manager has the option to sell or buy more or less arbitrary amounts of stocks in each period. The objective of the action might be a rebalancing of the portfolio or an adjustment of a hedge position. Assume an action operator A for an activity, that can be repeated arbitrarily often in instantaneous time. Let $A^\star$ be the optimal exercise of A with respect to utility U:

$$A^\star = A^{x^*} \tag{38}$$

Whereas x^* is the optimal operator power with respect to utility operator U.

$$x^* = \operatorname{argmax} U A^{x^*} \tag{39}$$

Hedging A typical application of the space continuous option is the optimal rehedge, where we can buy or sell an arbitrary number of stocks. The operator that buys one stock increases our deposit h by one and decreases our cash account by the current stock price S.

$$A = T_c^{-S} T_h \tag{40}$$

For an optimal hedge investment we have to apply the stock buying operator A with the optimal exponent.

$$A^\star = \left(T_c^{-S} T_h\right)^\star \tag{41}$$

For a function V that depends on c, h and some variables x the operator that buys n stocks can be solved explicitly.

$$A^n\, V(c, h, \mathrm{x}) = V(c - nS, h + n, \mathrm{x}) \tag{42}$$

The result of the optimal investment $A^\star$ is then obtained by maximizing the utility UA^nV over the number of bought stocks n.

5 Pricing via arbitrage

This section gives a quick introduction into arbitrage pricing to find the unique option price in a single step binomial tree. We will try to replicate the pay off of a call option precisely. We are looking for an investment strategy that yields the same final cash value as a call option.
In order to set up our strategy we write the usual operator sequence in chronological order from left to right. First we buy x stocks at price S, thus withdrawing xS from our account c. Then we by one option at a price y. Then we wait one period, sell all our stocks and exercise the option.

$$\Pi(\Theta) = \underbrace{T_c^{-\mathrm{x}S}\, T_c^{-\mathrm{y}}}_{\substack{\text{buy x stocks}\\ \text{at } S \text{ and } 1\\ \text{option at y.}}} \underbrace{\Theta}_{\text{wait}} \underbrace{T_c^{\mathrm{x}S}}_{\substack{\text{sell}\\ \text{stocks}}} \underbrace{\left\langle \begin{array}{l} \xrightarrow{\max} T_c^{S-K} \\ \to \ . \end{array}\right.}_{\text{exercise option}} \tag{43}$$

5.1 The process

Now we need to define what happens during the time of our inactivity. The time process consists of two effects. First, an interest rate of size rc is payed to the cash account c. And, second, two different branches are taken with probabilities p and $1 - p$. In the first case S is increased by $+S$, resulting in $2S$. The other possible outcome reduces S to its half.

$$\Theta = T_c^{rc} \left\langle \begin{array}{l} \xrightarrow{p} T_S^{+S} \\ \xrightarrow{1-p} T_S^{-\frac{1}{2}S} \end{array}\right. \tag{44}$$

Initial values The initial values of the process parameters are always inserted after the complete operator term is expanded. We will use the operator 大 to indicate this insertion. 大 is always the leftmost operator in a sequence, since the variables do no longer occur in the function after its application.

$$\text{大}\, V = V\Big|_{\substack{S=100 \quad c=0\\ K=150 \quad r=1/9}} \tag{45}$$

The operator 大 inserts the initial values and triggers a computational evaluation procedure, if necessary. Binomial tree models with short time horizons can normally be handled symbolically by computer algebra systems and thus allow the automated extraction of many implicit parameters.

5.2 Arbitrage-free price

In order to find the fair option price y, we have to find a hedge position x such that our final cash amount is zero under all conditions. Basically, we write our evaluation formula chronologically. First we fix the initial values, then go through our investment strategy Π and finally query our cash value V_c, with $V_c = c$ from (3). This operator sequence evaluates the expected amount π of account c after the completion of this strategy.

$$
\begin{aligned}
\pi &= 大\,\Pi(\Theta)V_c \qquad (46)\\
&= 大\,T_c^{-\mathrm{x}S}\,T_c^{-\mathrm{y}}\Theta T_c^{\mathrm{x}S}(c + \max(S-K,0))\\
&= 大\,T_c^{-\mathrm{x}S}\,T_c^{-\mathrm{y}}\Theta\,(c + \mathrm{x}S + \max(S-K,0))\\
&= 大\,T_c^{-\mathrm{x}S}\,T_c^{-\mathrm{y}}\big(c(1+r) + p\,(\mathrm{x}2S + \max(2S-K,0)) +\\
&\quad (1-p)\,(\mathrm{x}S/2 + \max(S/2-K,0))\,\big)\\
&= 大\big((c-\mathrm{x}S-\mathrm{y})(1+r) + p\,(\mathrm{x}2S + \max(2S-K,0)) +\\
&\quad (1-p)\,(\mathrm{x}S/2 + \max(S/2-K,0))\,\big)\\
&= p\left(50 + \frac{800}{9}\,\mathrm{x} - \frac{10}{9}\,\mathrm{y}\right) + (1-p)\left(-\frac{550}{9}\,\mathrm{x} - \frac{10}{9}\,\mathrm{y}\right)
\end{aligned}
$$

The final cash value is the same as has been derived in [5]. The difference is that this computation follows straight forward mathematical expansions. You should keep in mind that more realistic examples with multiple steps and additional assets quickly lead to algebraic results that can span several pages. A simple and compact calculus to command a computer algebra system is therefore essential. Fortunately, this result is short and several methods can be used to find the solution. The equation system that is to solve requires that the profit π is zero for all probabilities p.

$$\exists \mathrm{x}, \mathrm{y} :\ \forall p : \pi = 0 \qquad (47)$$

We can turn this into a finite system of equations, by inserting different values for p and verify the result. In this example we retrieve an option price of 55/3 and a hedge position of −1/3 stocks.

$$\widetilde{\mathrm{x}} = -\frac{1}{3}, \quad \widetilde{\mathrm{y}} = \frac{55}{3} \qquad (48)$$

The existence of a solution is not always guaranteed. The fact that we do have a valid single solution is due to the fact that we deal with a so called complete market, in which all options can be replicated by a unique stock trading strategy.

5.3 Equivalent martingale measure

A simplified and efficient method for finding the same price is done with the equivalent martingale measure. There exists a pseudo probability $\widetilde{p}$ for which the discounted stock price is a martingale, i.e. the discounted expected value tomorrow is equal to todays value.

$$\exists p : \frac{\Theta S}{1+r} = S \qquad (49)$$

The result is found easily.

$$\widetilde{p} = \frac{11}{27} \tag{50}$$

With the new value for p we can create a transformed process $\widetilde{\Theta}$ that evaluates the expected value under the equivalent martingale measure, where stock prices are expected to grow with the interest rate.

$$\widetilde{\Theta} = \Theta|_{p=\widetilde{p}} \tag{51}$$

This transformed operator can now be directly applied to the pay off structure of the option to compute the fair price, exactly as we did it in the Black&Scholes model (see 3.3).

$$大 \frac{\widetilde{\Theta} \max(S-K,0)}{1+r} = \frac{55}{3} \tag{52}$$

The evaluation operator 大 causes the computational system to switch to a numerical scheme after the full operator term was specified.

6 Example

In this final example we consider a portfolio manager who periodically rebalances her hedge portfolio. Our trader has an obligation to her customers in the form of a call option with strike 10. Suppose her mission was to optimally meet her obligation in either cash or stocks with a minimum squared distance. We do not work in complete markets, since we assume trading opportunities at discrete times and will later introduce market impact. Thus all hedges will bear at least some risk. Our function V contains the final wealth and its square.

$$V = \begin{pmatrix} V_a - V_{call} \\ -(V_a - V_{call})^2 \end{pmatrix} = \begin{pmatrix} c + Sh - \max(S-10,0) \\ -\big(c + Sh - \max(S-10,0)\big)^2 \end{pmatrix} \tag{53}$$

For a least square hedge we define the utility operator U as the second component of vector V.

$$UV = V_2 \tag{54}$$

We consider a single asset market where the price level S fluctuates according to a Brownian motion. The Lévy triplet for this process is $[\mu = 0, \sigma = 1, \nu = 0]$ and can be turned into operator form by equation (14).

$$\Theta = L^{-\frac{1}{2}u^2} \tag{55}$$

The numeric solution to the expected final amount of cash and the expected utility is done by the numerical evaluation scheme 大. This operator evaluates an approximation to the operator term ΘV and inserts the initial values for S, c and h. Depending on the chosen method 大 initializes a scenario in a Monte-Carlo simulation or retrieves the initial position in a PDE result.

$$大\Theta V = \Theta V\Big|_{\substack{S=10 \\ c=0.4 \\ h=0}} = \begin{pmatrix} 0 \\ -0.34 \end{pmatrix} \tag{56}$$

The first component tells us that the trader meets the expectation of her obligation precisely. Hence, the initial cash value of 0.4 is the expected value for the option. The low utility in the second component reveals the high risk inherent in holding the unhedged option.

6.1 Hedging activity

Now we want to see, if the utility can be increased by trading in the underlying stock with a reoptimization frequency of Δt. The operator that buys one stock subtracts the current stock price from cash account c and adds one stock the deposit h. The $\star$ indicates the optimal exponent.

$$\Pi(\Theta) = \left(\left(T_c^{-S}\, T_h\right)^{\star}\ \Theta^{\Delta t}\right)^{\frac{1}{\Delta t}} \tag{57}$$

According to our numeric results, the utility increases significantly with a portfolio rebalancing frequency of $\Delta t = 1/4$. The expected profit is still zero. The strategy produces no extra costs.

$$大\Pi(\Theta)V = \begin{pmatrix} 0 \\ -0.03 \end{pmatrix} \tag{58}$$

Complete market With our choice for the process Θ the risk of every obligation V can be reduced to zero by infinitely many rehedges. Markets governed by such processes are called complete markets [6].

$$\lim_{\Delta t \to 0} 大\Pi(\Theta)V = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{59}$$

6.2 Supply and demand

Due to the law of supply and demand real stock prices vary with the traded amount. Individual market participants will enter in the order book the prices at which they are willing to buy or sell. The more stocks we want to trade, the more people we have to satisfy and the worse is our price. We assume a linear order book in which every transaction has a price impact of κ per stock. The new strategy with $1/\Delta t$ rehedges, Π_I, considers the price impact on S.

$$\Pi_I(\Theta) = \left(\left(T_S^{\kappa}\, T_c^{-S}\, T_h\right)^{\star}\ \Theta^{\Delta t}\right)^{\frac{1}{\Delta t}} \tag{60}$$

Applied to our valuation function this reveals the expected final cash amount and the utility. The assumed market elasticity is $\kappa = 0.2$.

$$大\Pi_I(\Theta)V \begin{pmatrix} -0.05 \\ -0.04 \end{pmatrix} \tag{61}$$

Our trader is expected to loose 0.05 units of cash due to market friction. These are not transaction costs, since the selling operator is the exact inverse of the buying operator. The costs originate from procyclic trading and are gained by the anticyclic investor. If our trader wanted to reduce her loss then she had to take more risk. With respect to her quadratic utility function the presented values are optimal.

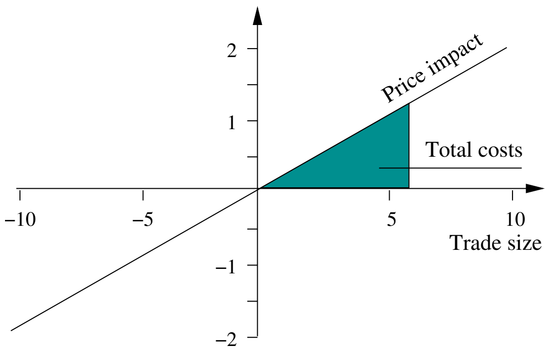


Figure 2: In the linear market impact model every traded stock shifts the price by $\kappa = 0.2$. The total cost, marked by the filled area, is proportional to the square of the trade size.

6.3 Market impact

Finally we might ask for the market impact of the hedging strategy on the stock price. The valuation function V is easily extended by additional components for the expected stock price S and its square.

$$\text{大}\Pi_I(\Theta)\begin{pmatrix} V \\ S \\ S^2 \end{pmatrix} = \begin{pmatrix} \Pi_I(\Theta)V \\ 10.117 \\ 103.5 \end{pmatrix} \tag{62}$$

The first interesting result is that the stock price S is expected to rise by 1.17%, which is due to the fact that we bought stocks for hedging purposes but did not necessarily sell them finally. Economists will refer to this phenomena as inflation, induced by a 40 cent increase of circulating cash and a corresponding overdemand on the stock exchange. The second parameter needs some treatment to reveal its information.

$$\tilde{\sigma} = \sqrt{\mathbb{E}(S^2) - \mathbb{E}(S)^2} = \sqrt{103.5 - 10.117^2} = 1.08 \tag{63}$$

We remember that the volatility σ was initialized to one in (55) and now increased to 1.08. The final result is that our hedging strategy is procyclic and increases the market volatility of the stock by 8%. This is a realistic value for very large investments.

7 Conclusion

This document introduced an operator term Π for the sequence of human investment activities. Three kinds of operations were considered to constitute the space of possible strategies. The first kind of occupation is inactivity. Whenever Θ occurs in an operator term it refers to a period of passive observation. During that time, external state variables may vary according to a stochastic process [6, 4] or as described by a partial differential equation [7]. The second possible activity is a transaction. The operator T initiates a deterministic effect on our parameter set. Typical instances are the transfer of goods or cash. The third and final operation models an option. Multiple operator terms $\Pi_1 \cdots \Pi_n$ can be offered as choices for further procedure. The decision criteria can be based on the current state and the expected values for each choice.

The operator term Π is written in chronological order from left to right and makes use of some mathematical concepts like the operator power for repeated actions. Solutions to risk measures and expected values can be evaluated directly with either a numerical or in some instances symbolic method.

References

[1] Fpml 4.0 specification. Technical report, www.fpml.com, 2004.

[2] G. Beylkin. On the representation of operators in bases of compactly supported wavelets. *SIAM Journal on Numerical Analysis*, 29(6):1716–1740, 1992.

[3] Simon Peyton Jones, Jean-Marc Eber, and Julian Seward. Composing contracts: an adventure in financial engineering, 2000.

[4] B. K. Ksendal and Bernt Oksendal. *Stochastic Differential Equations: An Introduction With Applications*. Springer-Verlag, 2001.

[5] S. M. Ross. *An Introduction to Mathematical Finance*. Cambridge university press, 1999.

[6] Steve E. Shreve. Stochastic calculus and finance, 1997.

[7] Domingo Tavella and Curt Randall. *Pricing financial Instruments The Finite Difference Method*. John Wiley & Sons, Inc., 2000.

[8] Rudi Zagst. *Interest Rate Management*. Springer Finance, 2002.