Convergence of the Ito integral - empirical analysis

March 11, 2004

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This paper is concerned with the solution of the Ito integral $\int W dW$. The numerical procedure is written using Multi Scale Calculus.

1 Brownian motion

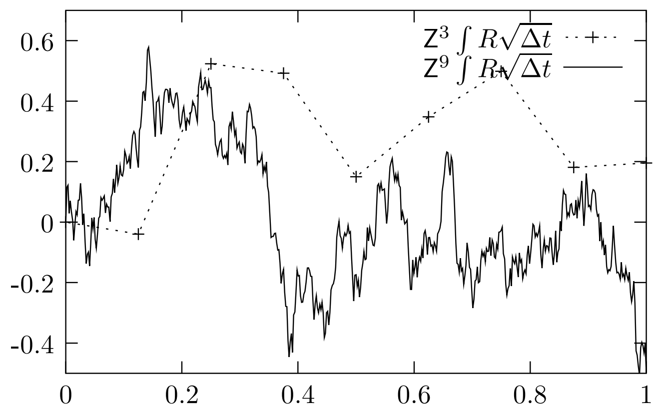
Random innovation operator R.

$$\text{大}R \backsim \mathcal{N}(0,1) \tag{1}$$

The easiest way to construct a sample of a Browinan motion is the integration of the noise process R.

$$\int \frac{R}{\sqrt{\Delta t}} \Delta t = \int R\sqrt{\Delta t} \tag{2}$$

The disadvantage of this approach is that only the initial value, here zero, can be controled and that further refined path's do not contain the one previously simulated.



Brownian motions at different resolutions

2 Brownian bridge

The Brownian bridge constructs a refinement of a discretely sampled Brownian motion that containes the course path and adds intermediate points. A refinement step of the bridge is evaluated by an interpolation of a course function Pf and increased by a random innovation QR.

$$\mathfrak{B}^\sigma f = Pf + \sqrt{\sigma} QR \tag{3}$$

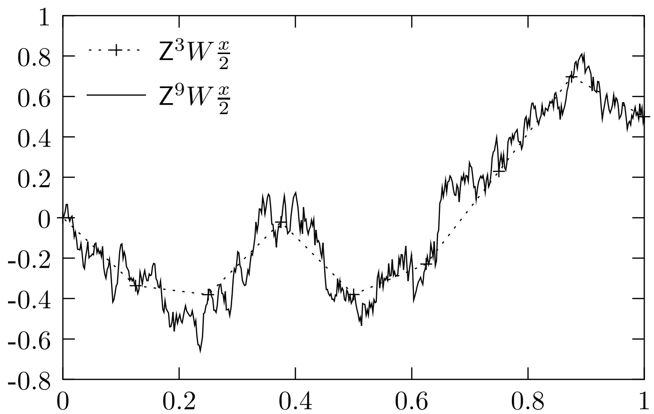
Whereas P and Q are wavelet prediction and refinement operators with coefficionts $\{0.5, 1, 0.5\}$ and $\{1, 0, 0\}$. A Brownian motion W has the following multi scale property:

$$\mathsf{Z} W^\sigma = W^{\sigma/2}(Pf + \sqrt{\sigma/2} QR) \tag{4}$$

Applying the W^1 to the initial function $\frac{x}{2}$ renders a Brownian motion that interpolates the linear function at all integer positions. In particular, it starts at zero and reaches 1/2 at time 1.

$$\text{大} W \frac{x}{2} = 0 \tag{5}$$

$$\text{大}\mathsf{T} W \frac{x}{2} = \frac{1}{2} \tag{6}$$



Brownian bridge construction at different resolutions

3 Ito integral

The Ito integral $\int_0^1 W dW$ is now written in multi scale calculus as

$$\int_0^1 W \left(\frac{\Delta W}{\Delta t} \right) \Delta t \tag{7}$$

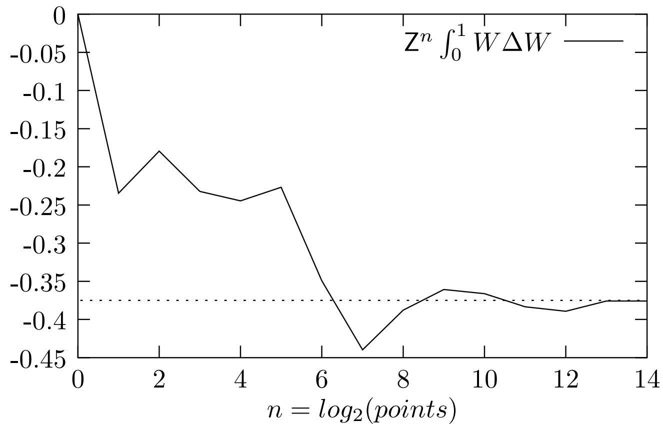
Canceling the Δt yields:

$$\int_0^1 W \Delta W \tag{8}$$

With the number of refinements Z^n, the integral converges to the Ito result.

$$\frac{1}{2} B(T)^2 + \frac{1}{2} T \tag{9}$$

Which in our example is -0.375.



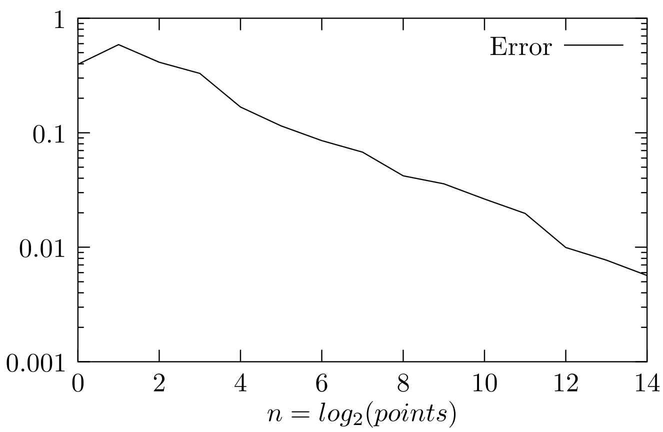
Convergence of the Ito integral

4 Convergence speed

The standard deviation of the error is

$$\mathsf{Z}^n \sqrt{\frac{1}{m-1}\int_0^m \left(\int_0^1 W\Delta W - (-0.375)\right)^2 \Delta\omega} \tag{10}$$

Deducing from ten ($m = 10$) samples, the integral seems to have linear convergence on the logarithmic scale.



Standard deviation of the error