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Dadim for Beginners

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Dedication

This work is dedicated to my wife Anna, who is about to finish her economy studies. The twelve chapters are designed to be understood with basic mathematical knowledge and explain the fundamential ideas behind Dadim and the new financial calculus. I hope that my excercises make her and all readers enjoy solving mathematical problems and that they help Anna to finish her studies successfully.

Work sheet 1

Operator notation

In our first session we will explore the nature of operators. Before we concern ourselves with the details of specific operators, we want to learn some of their basic properties. We always use capital letters to denote operators. Later we will see that each operator refers to an activity or an event. Sequences of operators are chronologically ordered lists of events that form a larger story.

Power The operator power is used as a short form for a repeated sequence of operators. It represents an event that occurs several times in a sequence.

$$A^n = \underbrace{AAA \cdots A}_{n} \tag{1}$$

Associativity We can drop and rearrange parentheses is our operator term.

$$(AB)C = A(BC) = ABC \tag{2}$$

Non commutativity This is warning, operators are not numbers!

$$AB \neq BA \tag{3}$$

Exercise

1. Simplify the following operator terms.
   (a) $BB\left(AA^2\right)^2 A^3 = B^2A^9$
   (b) $A(BA)AA$
   (c) $A(BA)(AB^5)B^{-1}$
   (d) $(AB)B(BA)A$

2. Write in a shorter form.
   (a) $ABABABABAB = (AB)^5$
   (b) $ABCABCABCCBCBCB$
   (c) $AAABBAAABBAAABB$
   (d) $AAAAABABABABABBBBB$

3. Write a chronological operator term for the following activities.

   W Working day
   R Relaxing day

   (a) For 6 weeks, work 5 days and relax on the weekends.
   (b) For 8 weeks, work on monday, thursday, saturday and sunday.
   (c) 3 weeks like (3a), then 2 weeks like (3b).

Work sheet 2

Transaction operator

The transaction operator T updates or modifies the value of a variable. It is usually associated with the transaction or transfer of goods. Sometimes it touches physical dimensions and is known as shift and translation operator.

$$T_x f(x) = f(x+1) \tag{4}$$

Writing a number n in the exponent of the operator is just an short form for the repeated application of the T operator.

$$\begin{aligned} T_x^n f(x) &= \\ T_x T_x \cdots T_x f(x) &= \\ f(x+1+1+\cdots+1) &= f(x+n) \end{aligned} \tag{5}$$

Note: The operator index tells you which variable to modify. The operator power contains the value, by which the variable changes. **Example:** the operator T_a^5 says: "Replace all instances of a with $a+5$".

Exercise

1. Compute:

   (a) $T_x^5 \left(T_y^2(xy)\right) = T_x^5 x(y+2) = (x+5)(y+2)$

   (b) $\left(T_x T_y^2\right)^2 (xy)$

   (c) $T_x^{10} T_z T_y^z (x^y)$

   (d) $\left(T_x^a T_a\right)^2 x$

2. What is the final value of cash account x if we receive one unit on working days ($W = T_x$) and spend 2 units when relaxing ($R = T_x^{-2}$).

   (a) What happens after one week with 5 working and 2 relaxing days?

   $$\begin{aligned} W^5 R^2 x &= T_x^5 \left(T_x^{-2}\right)^2 x \\ &= T_x^5 T_x^{-2}(x-2) \\ &= T_x^5 (x-4) \\ &= x+1 \end{aligned}$$

   Answer: after one week, the account x increases by one.

   (b) What is the value for work sheet 1 (3a)?

   (c) What is the value for Work sheet 1 (3b)?

   (d) What is the value for Work sheet 1 (3c)?

Work sheet 3

Interest rates and relative growth

A special case of the T operator occurs, when the exponent is a multiple of the index variable. Typically this is used for interest rate payments with rate r.

$$T_x^{rx} f(x) = f(e^r x) \tag{6}$$

This definition for the relative T operator was chose, such that it satisfies the following condition.

$$\begin{aligned} (T_x^{rx})^2 f(x) &= \\ f(e^r e^r x) &= \\ f(e^{2r} x) &= T_x^{2rx} f(x) \end{aligned} \tag{7}$$

Example: The operator T_a^{5a} says: "Replace all instances of a with ae^5".

Exercise

1. (a) $T_x^{xy}(x + y) = xe^y + y$
   (b) $T_x^{rx} T_x^r\, x$
   (c) $T_x^r T_x^{rx}\, x$
   (d) $T_x^y T_y^{xy}(xy)$

2. Assume that we get the interest rate $r = 0.1$ at the end of each day. The interest rate payment operator is $T_x^{0.1x}$. Thus, the working and relaying operator is now:
   $$\begin{aligned} W &= T_x\, T_x^{0.1x} \\ R &= T_x^{-2}\, T_x^{0.1x} \end{aligned}$$
   Recompute the examples of work sheet 2, exercise 2a,2b,2c,2d.

3. Equivalently we can compute the real value of our account x with a decay rate of $r = -0.1$ due to money inflation. Use $T_x^{-0.1x}$ instead of $T_x^{+0.1x}$ and recompute the examples of work sheet 2, exercise 2a,2b,2c,2d.

4. Over 5 years you get 100 per year (T_x^{100}). The interest rate of 5% is payed per year ($T_x^{0.05x}$). What is the final value of x.

Work sheet 4

Initial values

Sometimes it is required to insert initial values for our variables into an operator term. With the application of 大[1] all variables x are replaced with their initial values x_0.

$$\text{大} f(x) = f(x_0)$$

If initial values x_0 are not uniquely defined in the computation context, they might be explicitly specified underneath the 大 operator.

$$\underset{x=x_0}{\text{大}} f(x) = f(x_0)$$

Note: The evaluation with initial values is only performed, after all other operators have been applied. The operator 大 is therefore always the left most operator in a sequence. Sometimes it is implicitly assumed and not written.

Exercise

1. Assume the following initial values:

   $$\begin{aligned} a_0 &= 0 \\ b_0 &= 1 \\ c_0 &= \pi \end{aligned}$$

   (a) $\text{大} T_a(a+b+c) = (a_0 + 1 + b_0 + c_0) = 1 + 1 + \pi$
   (b) $\text{大} T_a T_b (ab)$
   (c) $\text{大} \left(T_a T_b^a T_c^b\right)^3 b$
   (d) $\text{大} T_a T_b^{ab} T_c^{bc}(c)$
   (e) $\text{大} \left(T_a T_a^{ab}\right)^3 a$

2. Recompute with $x_0 = 0$, the examples from worksheet 2, exercise 2a,2b,2c,2d and work sheet 3, exercise 2,3 and 4.

3. Evaluate

   (a) $\underset{c=4}{\text{大}} (2c) = 8$
   (b) $\underset{a=3}{\text{大}} \; \underset{b=10}{\text{大}} (b^a)$
   (c) $\underset{c=2}{\text{大}} \; \underset{b=c}{\text{大}} \; \underset{a=c}{\text{大}} (ab+c)$
   (d) $\underset{x=\sqrt{27}}{\text{大}} x^4$

4. Practice

[1] chinese for big, speak “da” as in lambda

Work sheet 5

Brownian motion

With the transaction operator T we are able to change a variable by a deterministic value. However, in the economy one has to deal with many random and unpredictable effects. The "blur" or "Brown" operator B adds a normally distributed value to the index variable and returns the expected value.

Polynomials The B operator can only be solved explicitly for a small set of arguments. The list below shows the operator effect up to a polynomial of order three.

$$B_x^n a = a \tag{8}$$
$$B_x^n ax = ax \tag{9}$$
$$B_x^n ax^2 = ax^2 + na \tag{10}$$
$$B_x^n ax^3 = ax^3 + 3nax \tag{11}$$

Exponential function

$$B_x^n \exp(ax) = \exp\left(ax + \frac{1}{2}na^2\right) \tag{12}$$

Linearity A very important property of the B operator is its linearity. When applied to a sum of functions, the B operator can be computed for each element individually.

$$B_x^n \left[f(x) + g(x)\right] = B_x^n f(x) + B_x^n g(x) \tag{13}$$

Exercise

1. Compute:
   (a) $B_x \left(ax^2 + bx + c\right) = B_x ax^2 + B_x bx + B_x c = a + ax^2 + bx + c$
   (b) $B_x^2 \left(5x^3 + 7x\right)$
   (c) $B_x^{0.5} \left(\exp(x) + x^2\right)$
   (d) $B_y B_x \exp(xy)$
   (e) $B_x T_x^{rx} x^2$
2. Write operator terms for the following events. Do not evaluate.
   (a) Variable x is first increased by 5% and then a random number with variance 1 added.
   Solution: $T_x^{0.05x} B_x$
   (b) First increase account x by 10, then vary interest rate r with variance 0.01. Then pay a relative interest rate r onto account x.
   (c) Repeat 10 times: Increase x by 1, add a random variable with variance 0.5.

Work sheet 6

Statistical measures

As we have seen, the operators T and B can describe deterministic and stochastic impacts on our variables. Now we want to extract their statistical properties. Assume, that Θ is a sequence of operators.

Variance

$$\mathrm{Var}(x) = \Theta\left(x^2\right) - (\Theta x)^2 \tag{14}$$

Standard deviation

$$\mathrm{stdev}(x) = \sqrt{\mathrm{Var}(x)} \tag{15}$$

Covariance

$$\mathrm{Cov}(x,y) = \Theta(xy) - (\Theta x)(\Theta y) \tag{16}$$

Correlation

$$\mathrm{Cor}(x,y) = \frac{\mathrm{Cov}(x,y)}{\sqrt{\mathrm{Var}(x)}\sqrt{\mathrm{Var}(y)}} \tag{17}$$

Exercise

1. Compute the Variance in x of the following processes.

   (a) $\Theta = B_x^\sigma T_x^5$
   Solution:

$$\begin{aligned}
\Theta\left(x^2\right) - (\Theta x)^2 &= \\
B_x^\sigma T_x^5\left(x^2\right) - \left(B_x^\sigma T_x^5 x\right)^2 &= \\
B_x^\sigma\left((x+5)^2\right) - \left(B_x^\sigma(x+5)\right)^2 &= \\
\left((x+5)^2+\sigma\right) - (x+5)^2 &= \sigma
\end{aligned}$$

   (b) $\Theta = B_x^a B_x^b$
   (c) $\Theta = B_y^\sigma T_x^{2y}$
   (d) $\Theta = T_a^5 B_x^a$
   (e) $\Theta = B_a B_x^a$

2. Compute the covariance and the correlation between x and y in the following processes.

   (a) $\Theta = B_x B_y$
   (b) $\Theta = T_x^y B_x B_y T_x^{-y}$
   (c) $\Theta = T_x^{ay} B_x B_y$
   (d) $\Theta = B_x T_y^{x^3}$

Work sheet 7

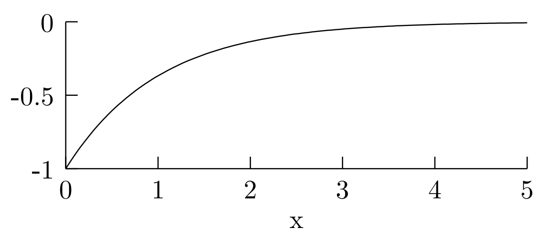
Intertemporal optimization

In this work sheet we will develop a complex sequence of actions with intertemporal optimization decisions. Assume a stock price x varies according to a stochastic process Θ. The variable c represents our cash account and h contains the number stocks in our deposit. We start with zero wealth, but an unlimited credit line.

variable	description	initial value
c	cash value	$c_0 = 0$
h	number of stock in deposit	$h_0 = 0$
x	stock price	$x_0 = 1$

Our utility function u depends on our wealth, which is the sum of cash c and the value of our stock deposit hx.

$$u = -\exp\left(-(c + hx)\right) \tag{18}$$



The utility that results from holding one stock with price x.

$$-\exp(-x)$$

The stock price x is governed by the process Θ with variance 0.2 and drift 0.1.

$$\Theta = B_x^{0.2} T_x^{0.1} \tag{19}$$

For buying n stocks we use the buying operator A, that subtracts nx from the cash account and adds n to the deposit.

$$A = T_c^{-nx} T_h^{n} \tag{20}$$

Now we compute the optimal investment n and the resulting utility $A\Theta u$. The sequence reads chronologically in time. First we invest in stocks A, then wait for the stocks to move Θ and finally evaluate the utility u.

$$A\Theta u = A\Theta\left(-\exp(-c - hx)\right) \tag{21}$$

$$= A\left(-\exp\left(-c - \frac{1}{2} - \frac{h}{10} - hx + \frac{h^2}{50}\right)\right)$$

$$= -\exp\left(-c - \frac{1}{2} - \frac{n}{10} - \frac{h}{10} - hx + \frac{n^2}{50} + \frac{nh}{25} + \frac{h^2}{50}\right) \tag{22}$$

The final utility depends on the number of bought stocks n. To find the optimal deposit, we differentiate (22) by n.

$$-\left(-\frac{1}{10} + \frac{n}{25} + \frac{h}{25}\right)\exp\left(-c - \frac{n}{10} - \frac{h}{10} - hx + \frac{n^2}{50} + \frac{nh}{25} + \frac{h^2}{50}\right) \tag{23}$$

Equating the derivative to zero, reveals the optimal stock investment $n^\star$.

$$n^\star = \frac{5}{2} - h \tag{24}$$

The result is, that we should hold 5/2 stocks, which is achieved by buying 5/2 minus h, the number we already possess. Inserting the optimal n back into (22) gives us the expected utility after one time step.

$$\tilde{u} = -\exp\left(-\frac{5}{8} - hx - c\right) \tag{25}$$

We can return with $\tilde{u}$ to equation (21) and insert it for u. Repeating the optimization gives us the expected utility after the next time step.

Exercise

1. Given the initial values, what is the expected utility after one, two and three time steps? Hint: you need to evaluate the optimization after each time step. (Results: -0.88, -0.78, -0.69)

2. Introduce interest payments to the account and compute the expected utility after one time step. (Result: -0.9997)

$$\Theta = B_x^{0.2} T_x^{0.1} T_c^{0.1c}$$

3. Repeat the computations with

$$\begin{aligned} \Theta &= B_x^{0.05} T_x^{0.1} \\ u &= -(c + hx - 1)^2 \end{aligned}$$

(Results: -0.2, -0.04, -0.008)

4. For all examples compute the expected wealth $(hx+c)$, under the optimal investments.

Results:

1: 0.25, 0.5, 0.75

2: .00067

3: 0.8, 0.96, 0.992

Work sheet 8

Geometric Brownian motion

In analogy to the relative T operator we can generate a blur operator B with an exponent that is proportional to the square of the index variable. The resulting process is a geometric Brownian motion, with a standard deviation proportional to the value. This process is often used for stock prices and economic data, that varies with an intensity that rises with the absolute price level. Unfortunately this operator can only be solved analytically for a small set of operands.

$$B_x^{(ax)^2} x^n = \exp\left(\frac{1}{2}n(n-1)a^2\right) x^n \tag{26}$$

$$B_x^{(ax)^2} \ln(x) = \ln(x) - \frac{1}{2}a^2 \tag{27}$$

Exercise

1. Compute
   (a) $B_x^{ax^2} B_x^a x^2$
   (b) $B_x^a B_x^{ax^2} x^2$
   (c) $T_x^2 B_x^{ax^2} \ln(x)$
   (d) $B_x^{ax^2} T_x^2 \ln(x)$
2. The Black-Scholes model for stock prices can be written as

   $$\Theta = T_x^{rx} B_x^{\sigma^2 x^2}.$$

   Compute the expectation of the following functions under the Black-Scholes model
   (a) Θx
   (b) $\Theta^2 x$
   (c) Θx^2
   (d) $\Theta^2 x^2$
   (e) $\mathrm{Var}(x)$
   (f) $\Theta \ln(x)$
   (g) $\Theta^2 \ln(x)$

Work sheet 9

Binomial trees

As we have seen, the operator B is often extremely difficult to evaluate. Solutions exist only for a small selection of functions. A simple, but good approximation to the B operator can be achieved by the binomial tree model. Instead of considering a normally distributed variable we compute the expectation of only two possible outcomes, one that goes up and one that goes down.

$$\begin{tikzcd}[row sep=small] & T_x^{\sqrt{a}} \\ B_x^a \approx \arrow[ur, no head, "1/2"] \arrow[dr, no head, "1/2"'] & \\ & T_x^{-\sqrt{a}} \end{tikzcd}, \qquad a \in \mathbb{R} \tag{28}$$

Thus we use the operator $\widetilde{B}$ for the approximation operator.

$$\widetilde{B}_x^a f(x) = \frac{1}{2}\left(T_x^{\sqrt{a}} f(x) + T_x^{-\sqrt{a}} f(x)\right) \tag{29}$$

Exercise

1. Compute:

   (a) $\widetilde{B}_x^{\sigma^2} x^2 = \frac{T_x^{\sigma} x^2 + T_x^{-\sigma} x^2}{2} = (x+\sigma)^2 + (x-\sigma)^2 = x^2 + \sigma^2$

   (b) $\left(\widetilde{B}_x^{\sigma^2}\right)^2 x^2$

   (c) $\left(\widetilde{B}_x^{\sigma^3}\right)^2 5x^3$

   (d) $\widetilde{B}_x^4 f(x)$

   (e) $\widetilde{B}_x^{\pi^2} \sin(x)$

   (f) $\widetilde{B}_x^{\pi^2/4} \sin(x)$

Work sheet 10

Geometric binomial trees

The approximation to the geometric Brownian motion is done similarly to (28). The only difference is that we use pseudo probabilities p and q instead of $1/2$.

$$B_x^{ax^2} \approx \begin{array}{l} \overset{p}{\nearrow} T_x^{x\sqrt{a}} \\ \underset{q}{\searrow} T_x^{-x\sqrt{a}} \end{array}, \qquad a \in \mathbb{R} \tag{30}$$

The approximation operator $\widetilde{B}$ is straight forward.

$$\widetilde{B}_x^{ax^2} f(x) = pT_x^{x\sqrt{a}} f(x) + qT_x^{-x\sqrt{a}} f(x) \tag{31}$$

Exercise

1. Compute p and q, such that
$$\widetilde{B}_x^{ax^2} x = x$$
Solution:
$$p = \frac{1}{\exp\left(\sqrt{a}\right) + 1}, \quad q = 1 - p$$
This p and q are also called risk neutral pseudo probabilities. They are used for all further instances of $\widetilde{B}$.

2. Compute
   (a) $\widetilde{B}^{\sigma^2 x^2} x^2$
   (b) $\widetilde{B}^{\sigma^2 x^2} \ln(x)$
   (c) $\widetilde{B}^{\sigma^2 x^2} \max(x - 1, 0)$

3. The variance of geometric Brownian motions has to be evaluated differently. Compute
$$\mathrm{Var}\big(\ln(x)\big)$$
for the following processes. Assume $x_0 = 1$.
   (a) $\Theta = \widetilde{B}_x^{x^2}$
   (b) $\Theta = T_x^{0.02x} \widetilde{B}_x^{0.1x^2}$
   (c) $\Theta = T_x^{xy} \widetilde{B}_x^{0.01x^2} \widetilde{B}_y^{0.02y^2} T_x^{-xy}$
   For this process compute also: $\mathrm{Var}\big(\ln(y)\big)$, $\mathrm{Cov}\big(\ln(x), \ln(y)\big)$ and $\mathrm{Cor}\big(\ln(x), \ln(y)\big)$

4. The following process is the Black-Scholes model for stock prices, as it is used for option pricing. The operator contains a factor e^{-r} for discounting the final payment.

$$\Theta = e^{-r}\, T_x^{rx}\, B_x^{\sigma^2 x^2}$$

For good numerical approximation we can split the process into n smaller time steps and use $\widetilde{B}$ instead of B.

$$\widetilde{\Theta}^{\frac{1}{n}} = e^{\frac{-r}{n}}\, T_x^{\frac{rx}{n}}\, \widetilde{B}_x^{\frac{\sigma^2 x^2}{n}}$$

Approximate the following options with 2 and 4 time steps. $r = 0.1$, $\sigma = 0.4$ and $x_0 = 100$.

(a) $\Theta^4 \max(x - 100, 0)$

(b) $\Theta \max(x - 80, 0)$

(c) $\Theta^2 \max(100 - x, 0)$

Work sheet 11

Ho&Lee interest rate model

The forward rate for time T is defined by a $f(T)$. We start with a given curve $f_0(T)$ and add random effects in form of parallel shifts a and rotations b.

$$f(T) = f_0(T) + a + bT \tag{32}$$

The initial values for a and b are both zero. We assume that a time counter t starts with zero as well.

$$a_0 = 0, \quad b_0 = 0, \quad t_0 = 0 \tag{33}$$

The process operator Θ for the Ho&Lee term structure model is given by a sequence of five operators. A random effect is added to a. A drift correction is performed in the variables a and b. Then the value of a unit of wealth € is updated and finally a time counter t is increased by one.

$$\Theta = B_a^{\sigma^2} \, T_a^{\mu_a} \, T_b^{\mu_b} \, T_{€}^{-f(t)€} T_t \tag{34}$$

The unit of wealth € can be treated like a normal variable. Economically it presents todays discounted value of one unit of cash.

Exercise

1. Compute μ_a and μ_b such that

   $$\Theta€ = €\exp\big(-f(0)\big)$$

   and

   $$\Theta^2€ = €\exp\big(-f(0) - f(1)\big)$$

   Solution:

   $$\mu_a = \mu_b = \frac{1}{2}\sigma^2$$

2. For $\sigma = 0.1$ and $f_0(M) = 0.1 + 0.01M$ compute

   (a) $\mathrm{stdev}\big(f(3)\big)$

   (b) $\mathrm{stdev}(€)$

   (c) $\Theta€$

   (d) $\Theta^2€$

   (e) $\Theta^3€$

   (f) $\Big(\Theta T_c^{c€}\Big)^3 c$

   (g) $\mathrm{cor}\big(f(0), f(1)\big)$

   (h) $\mathrm{cor}\big(f(0), €\big)$

3. The operator Θ represents one time step of passive observation. In sequence of operators Π we represent a complete investment and compute the expected profit and the variance of an investment.

   (a) Pay 90€, wait 10 periods and receive 100€.
   Solution:
   $$\Pi(\Theta) = T_x^{-90€}\Theta^{10}T_x^{100€}$$
   Profit $= \Pi(\Theta)x = ...$
   Variance $= \Pi(\Theta)x^2 - (\Pi(\Theta)x)^2 = ...$

   (b) Pay 500€ and receive over ten periods a coupon of 100€.

   (c) Repeat 3 times: wait (Θ), receive 1€, wait, pay 1€.

   (d) Assume $W = \Theta T_x$ and $R = \Theta T_x^{-2}$. Compute the variance of the strategy in worksheet 2, exercise 2a,2b,2c and 2d

Work sheet 12

Operator definition

So far we have seen the solutions for some specific cases of operators, operator exponents and arguments. In this final section we will derive the general definition for the operators T and B.

Shift operator

The shift or transaction operator is defined by the solution of the following partial differential equation. The exponent of T is a multiple of a function $tf(x)$ and is applied to a function $g(x)$. The result, derived by x time $f(x)$ equals the result derived by t.

$$\frac{d}{dt}T^{tf(x)}g(x) = f(x)\frac{d}{dx}T^{tf(x)}g(x) \tag{35}$$

Example:

$$T^{tx}x = \exp(t)x$$

Proof:

$$\frac{d}{dt}\exp(t)x = \exp(t)x = x\frac{d}{dx}\exp(t)x$$

Blur operator

The blur or Brown operator B solves the heat equation.

$$\frac{d}{dt}B^{tf(x)}g(x) = \frac{f(x)}{2}\frac{d^2}{dx^2}B^{tf(x)}g(x) \tag{36}$$

Example:

$$B^{tx^2}\ln(x) = \ln(x) - \frac{t}{2}$$

Proof:

$$\frac{d}{dt}\left[\ln(x) - \frac{t}{2}\right] = -\frac{1}{2} = \frac{x^2}{2}\frac{d^2}{dx^2}\left[\ln(x) - \frac{t}{2}\right]$$

Exercise

1. Verify that,

   (a) $T_x^{t\sqrt{x}} f(x) = f\left(\frac{t^2}{4} + t\sqrt{x} + x\right)$

   (b) $T_x^{t/x} f(x) = f\left(\sqrt{2t + x^2}\right)$

   (c) $T_x^{t\exp(x)} f(x) = f\left(-\ln\left(\exp(-x) - t\right)\right)$

   (d) $B_x^{tx^2} x^n = x^n \exp\left(\frac{1}{2}n(n-1)t\right)$

   (e) $B^t x^5 = x^5 + 10tx^3 + 15t^5$

2. Find solutions for (advanced students only):

   (a) $T_x^{a+bx} f(x)$

   (b) $T_x^{f(x)} x$

   (c) $B_x \exp\left(-x^2\right)$