An introduction to Theta-calculus

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Abstract

Theta-calculus is a mathematical calculus for the description of sequential processes, financial contracts and multiperiod strategies in game theory. This calculus allows the explicit notation of all trading strategies and financial products, that can currently not be written in an explicit mathematical form. All kinds of contracts, strategies and multiperiod games can then be captured in terms of their quantitative implications by a vocabulary of three basic effects: waiting, transacting and deciding. Each elementary activity is represented by an operator, that can be interpreted in an operator sequence as a chronologically ordered list of events. The operator term also represents an explicit formula for the evaluation of the respective strategy's final result. Theta-calculus is especially useful for the notation of financial products, many of which can currently not be represented explicitly. Thus, it places itself as an alternative to the methods of stochastic analysis as well as some technical standards, that aim for a representation of sequential processes and financial products.

0 Introduction

Quantitative finance is one of the most actively researched scientific fields dealing with processes. Many aspects of stochastic and deterministic processes as well as decision and game theories are found in finance. Despite a vast background of mathematical theory and concepts, prevailing financial calculus is unable to formalize one of the most fundamental aspects of trading and market analysis. In fact, financial calculus lacks an explicit mathematical notation for human trading activities and many financial products. Starting with the American option, everything that requires non trivial intertemporal decisions or optimizations can not be represented in a reasonable form, from which the product evaluation can be derived algebraically.
Elaborate contract types and investment objectives are typically specified in prose form only and typically use specific terminology that is hard to interpret by an uninvolved. Such representations are difficult to evaluate mathematically and have to be translated into formulas and computer code individually. Those who feel inclined to pursuit greater generality are mostly struck by an inflation of parameters and mathematical concepts.

In order to meet the industry demand for a technical portfolio representation there has been development in the extension of existing programming dialects. Most notable results are *MLFi* [3], based on the functional programing language Caml, and the XML-standard *Fpml* [1]. However, neither provides a mathematical framework for the derivation of theoretical properties. Their vocabulary is huge and gets still extended. Finally, there is a large distance from product representation to the evaluation procedure, which raises the fear of model ambiguity and inconsistency.
This document suggests an explicit and mathematically precise operator notation for trading strategies and financial derivatives. The notation is based on the foundations of operator theory and introduces a vocabulary of three operators: waiting, transacting and deciding. Each of the possible activities is represented by an operator, which is written in a chronological list to express a sequence of trading activities. The notation thus provides explicit expressions for contract details and trading strategies including embedded options, minimum guarantees, event triggers and non-delta hedges. Furthermore the notation yields a rather explicit procedure to extract its statistical properties, which can be performed by a computer algebra system or by a numerical scheme that is directly derived from the operator sequence.

1 The Theory

The theory of theta-calculus can be explained very compactly. This section shows the three elementary operators that allow the specification of all trading strategies and demonstrate the mathematical usage for the derivation of algebraic results. The following section two features examples and deepens the understanding of theta-calculus in the context of various existing models.

1.1 The three elementary operators

The basic idea of theta-calculus is a notation of all strategies, games and portfolios in terms of three operators referring to the activities: waiting, transacting and deciding. These activities are written by the mathematical symbols Θ, T and a branch operator. [2]

Operator theory Every operator is a function mapping a real valued function on an other real valued function. We will repeat the basic algebraic laws for two operators O_1 and O_2.

$$O_1, O_2 : (\mathbb{R}^n \to \mathbb{R}^m) \to (\mathbb{R}^n \to \mathbb{R}^m) \tag{1}$$

The operators are written without parentheses and may occur in operator sequences, where they are evaluated from right to left.

$$O_1 O_2 := f \mapsto O_1\left(O_2(f)\right) \tag{2}$$

For the linear combination of operators, the distributivity law applies.

$$O_1 + O_2 := f \mapsto O_1 f + O_2 f \tag{3}$$

1.1.1 Theta operator

The Θ operator is the first in our series of elementary operators and refers to a time step without activity or of passive observation. Whenever Θ occurs in a sequence of event operators the economic state is propagated by the "outside world" in a possibly random manner. The process operator Θ can be taken to the power of Δt to describe none unit time steps.

$$\Theta^{\Delta t} = \text{“Wait time } \Delta t\text{”} \tag{4}$$

Θ is mathematically defined as the expected value of the argument function f applied to tomorrow's state $X(t + \Delta t)$ given a current state $x = X(t)$.

$$\Theta^{\Delta t} f(x) := \mathbb{E}\left[f\big(X(t+\Delta t)\big)\big|X(t) = x\right] \tag{5}$$

The operator always corresponds to a Markovian process, or one that can be turned into Markovian form. The discussion of further properties and the expression of standard processes is continued in section 2.2

1.1.2 Transfer operator

The transaction operator T increases a process parameter by a deterministic amount. It is used for the transfer of goods or assets between accounting variables and to apply deterministic impacts on any process variable.

$$T_x^{\Delta x} = \text{“transfer } \Delta x \text{ units onto variable } x\text{”} \tag{6}$$

Mathematically, the operator replaces every instance of the index variable with the variable plus the one, or an operator exponent Δx if applied more than once. Applied to a function f that depends on the value of the parameter x the T_x operator is defined as follows:

$$T_x^{\Delta x} f(x) := f(x + \Delta x) \tag{7}$$

The exponent Δx may functionally depend on x.

1.1.3 Decision operator

An option is defined by the alternatives among which can be selected and by the entity that does decide. Feasible choices are specified by two event operators O_1 and O_2. Depending on the choice, one of the optional action sequences determines the remaining sequence after the option expired. The deciding entity is characterized by her choice condition C.

$$\begin{matrix} \overset{C}{\nearrow} O_1 \\ \underset{1-C}{\searrow} O_2 \end{matrix} := CO_1 + (1-C)O_2 \tag{8}$$

The choice condition C is an operator. In the most common cases the function itself carries the information on which choice is preferred. We can choose the more valuable scenario the with the choice condition $C_{\max}$.

$$C_{\max} = 1_{O_1 > O_2} \tag{9}$$

1.2 Evaluation sequence

In order to derive any information from Θ-calculus terms one must determine the initial state of the process variables, run through the defined strategy and ask a question about the final state.

1.2.1 Initial values

The Chinese character[1] 大 is used as an operator to insert initial values. It is applied from the left hand side an is always the first operator in a sequence. It simply replaces every remaining occurrence of a state variable with its initial value.

$$\underset{x=X_0}{大} f(x) := f(X_0) \tag{10}$$

Alternatively, this operator may be written with the bar operator.

$$\underset{x=X_0}{大} f := f\Big|_{x=X_0} \tag{11}$$

1.2.2 Final question

After running through the strategy we can ask for the expected value of any function of state variable or the expected value of function of the state variables $f(x)$.

$$\underset{x=X_0}{大} \text{...strategy...} f(x) \tag{12}$$

[1] pronounced “da” in mandarin

Example For the usage of such an evaluation sequence we investigate a simple investment. Suppose sequence of three effects. First you pay 90 in order to enter the investment. Then you have to pay interest on the dept you might have taken. Finally you receive a revenue of 100. This sequence can be written chronologically with the corresponding operators.

$$\underbrace{T_c^{-90}}_{\substack{\text{pay}\\90}}\ \underbrace{T_c^{rc}}_{\substack{\text{pay}\\\text{interest}}}\ \underbrace{T_c^{100}}_{\substack{\text{get}\\100}} \tag{13}$$

We begin with an initial account value of $c = 0$ and ask for the final balance c after the strategy completed.

$$\begin{aligned} &\underset{c=0}{大}\, T_c^{-90} T_c^{rc} T_c^{100} c = \\ &\underset{c=0}{大}\, T_c^{-90} T_c^{rc} (c + 100) = \\ &\underset{c=0}{大}\, T_c^{-90} (c(1 + r) + 100) = \\ &\underset{c=0}{大}\, (c - 90)(1 + r) + 100 = \quad 100 - 90(1 + r) \end{aligned} \tag{14}$$

The result is also known the present value of that investment, based on which it can be valued and compared to other possible strategies.

2 Examples

The basic concept behind theta-calculus has now been defined. We are able to write sequences of events and strategies as operator terms. Furthermore, we can evaluate statistical measures for the final result of all state variables. In order to fully comprehend the power and the flexibility of this approach we need to give more examples. Presented examples range from the expression standard financial products and processes and extend to the representation of arbitrage pricing, some aspects of game theory and sophisticated hedging strategies in incomplete markets.

2.1 Financial products

A main goal of Θ-calculus was the ability to describe complex contract types, game rules and trading strategies. We will now derive mathematical terms for the most common financial strategies and products. We will see how to incorporate outstanding events, embedded options, sensitivities to random events and the room for further trading activity.
Prevailing mathematical methods do not allow for the definition of complex derivative securities or trading strategies. Consequently all publications use text

and prose form descriptions to define the product or the strategy of question. The inability of proper notation also leads to an incomplete computational representation. Various inferior technical standards were introduced by the industry, such as Fpml, but none has a mathematical framework. Theta-calculus solves this shortcoming, since it provides a mathematically precise and easy to interpret description of financial contracts and strategies.

2.1.1 Fixed income

The term fixed income refers to financial products with payments that are induced deterministically by the process state. This does not necessarily yield a deterministic result, since state parameters may fluctuate randomly during a Θ step. We will generally assume, that the state variable c is our account balance and all funding and all payments are taken from or payed into this account c.

Bond A simple bond deal looks like this:

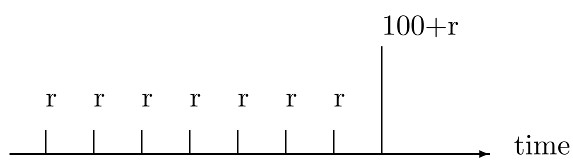
$$T_c^{-90} \Theta T_c^{100} \tag{15}$$

This corresponds to a chronological sequence of activities. Firstly, withdraw 90 units from your cash account c. Then, wait one period (maturity is one). Finally add 100 units onto your account.

Coupon bond A coupon bond with maturity 10 and coupon rate 5 looks like that:

$$\left(\Theta T_c^{5}\right)^{10} T_c^{100} \tag{16}$$

Read: Wait and receive 5. Repeat 10 times. Then receive 100.
The cash flow of a coupon bond is visualized in the plot below.



Swap A swap is an arrangement in which two parties repeatedly exchange assets at predefined conditions. In the case of currency swaps they exchange certain amounts in different currencies over a certain period. For interest rate swaps one pays a fixed size while the other pays a state dependent number. The swap investment Π_s simultaneously transacts r_1 in one direction and r_2 to opposite side over M periods.

$$\Pi_s(\Theta) = \left(\Theta\, T_c^{-r_1}\, T_c^{r_2}\right)^M \tag{17}$$

2.1.2 European type options

An European type option is a contract with the right to choose between predefined alternative investments A and B after option maturity. The operator sequence is waiting M times and finally deciding.

$$\Theta^M \begin{array}{l} \nearrow A \\ \searrow B \end{array} \tag{18}$$

Bond option Combining the option and the transaction operator we can specify our first interest rate derivative. The operator term below describes an option on a zero coupon bond. Again, the term reads chronologically from left to right. First, we wait a time m and then choose between two scenarios. In the first scenario we first pay a strike price K and wait for the underlying maturity M until we receive a final redemption of 100. The second scenario is the identity operator and refers to no transaction.

$$\Theta^m \begin{array}{l} \overset{C}{\nearrow} T_c^{-K} \Theta^M T_c^{100} \\ \underset{1-C}{\searrow} . \end{array} \tag{19}$$

Asian option The Asian option, as denoted below, is easily read from left to right. First we initialize an additional accounting variable a to have an initial value of 0. Then we repeat n times a time step that waits one period and then adds the current stock price S to a. Finally we can choose to receive the difference of the average stock price a/n and the strike price K as a cash payment.

$$\underset{a=0}{\text{大}} \left(\Theta T_a^S\right)^n \begin{array}{l} \overset{C}{\nearrow} T_c^{a/n-K} \\ \underset{1-C}{\searrow} . \end{array} \tag{20}$$

2.1.3 Time continuous options

The time continuous option, also referred to as American option, is an option with the continuous right of exercise. Over a certain time span M the holder of the option has the continuous right to leave the product path and branch into the optional product Π_X. The option requires a time process that can be infinitely subdivided and an option that is exercisable after each infinitely small time step. The American option is defined by the limit of the number of time steps to infinity and the length of each step Δt to zero.

$$\lim_{\Delta t \to 0} \left(\Theta^{\Delta t} \begin{array}{l} \overset{C}{\nearrow} \Pi_X \\ \underset{1-C}{\searrow} . \end{array} \right)^{M/\Delta t} \tag{21}$$

As an alternative to the limit function we can write the same term with the differential form dt replacing the discrete time step Δt.

$$\left(\Theta^{dt} \begin{array}{l} \overset{C}{\nearrow} \ \Pi_X \\ \underset{1-C}{\searrow} \ . \end{array} \right)^{M/dt} \tag{22}$$

American stock options The American stock option consists of a cash transaction for the choice Π_X, which when executed ends the investment. First, the amount $S - K$ is transfered. Then the value function f ends the operator term. The function can be considered as a constant operator that evaluates f regardless of what function it is applied to.

$$\Pi_X = T_c^{S-K} f \tag{23}$$

2.1.4 Optimal trades

Optimal trades are options, as they allow freely chosen amounts to be bought and sold at market prices. However, they require continuous values to fully describe the selected action. A portfolio manager has the option to sell or buy more or less arbitrary amounts of stocks in each period. The objective of the action might be a rebalancing of the portfolio or an adjustment of a hedge position. Assume an action operator A for an activity, that can be repeated arbitrarily often in instantaneous time. Let $A^\star$ be the optimal exercise of A with respect to utility U:

$$A^\star = A^{x^*} \tag{24}$$

Whereas x^* is the optimal operator power with respect to utility operator U.

$$x^* = \operatorname{argmax} U A^{x^*} \tag{25}$$

Hedging A typical application of the space continuous option is the optimal rehedge, where we can buy or sell an arbitrary number of stocks. The operator that buys one stock increases our deposit h by one and decreases our cash account by the current stock price S.

$$A = T_c^{-S} T_h \tag{26}$$

For an optimal hedge investment we have to apply the stock buying operator A with the optimal exponent.

$$A^\star = \left(T_c^{-S} T_h\right)^\star \tag{27}$$

For a function f that depends on c, h and some variables x the operator that buys n stocks can be solved explicitly.

$$A^n f(c, h, \mathrm{x}) = f(c - nS, h + n, \mathrm{x}) \tag{28}$$

The result of the optimal investment $A^\star$ is then obtained by maximizing the utility $UA^n f$ over the number of bought stocks n.

2.2 Financial processes

The process of financial parameters may also be referred to as the passive process. If no action is taken, only the Θ operator is in place and moves the process parameter. Before we start setting up various process operators we will first check some general properties of this Θ.
The process operator $\Theta^{\Delta t}$ is defined as the expected value of the argument function f applied to tomorrow's state $X(t+\Delta t)$ given a current state $\mathrm{x} = X(t)$. The operator always corresponds to a Markovian process, or one that can be turned into Markovian form.

$$\Theta^{\Delta t} f(\mathrm{x}) := \mathbb{E}\left[f\big(X(t+\Delta t)\big) \Big| X(t) = \mathrm{x} \right] \tag{29}$$

Another version of the same operator is given by the probability density $p_{\Delta t}(\mathrm{x}, \mathrm{y})$ for the state to travel from x to y within one time step. Explicit formulas for this density are derived below for the most common processes.

$$\Theta^{\Delta t} f(\mathrm{x}) = \int_{\mathbb{R}} p_{\Delta t}(\mathrm{x}, \mathrm{y}) f(\mathrm{y}) d\mathrm{y} \tag{30}$$

Lemma: Multiple applications of the process $\Theta^{\Delta t}$ evaluate the same expected value as (29), but with the new time horizon in the exponent.

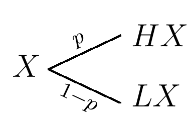
$$\left(\Theta^{\Delta t}\right)^n = \Theta^{n\Delta t} \tag{31}$$

Proof: The correctness of the operator power rule (31) is verified through repeated application of Θ according to its definition (29).

$$\begin{aligned}
(\Theta^{\Delta t})^n f(\mathrm{x}) &= (\Theta^{\Delta t})^{n-1}\, \mathbb{E}\left[f(X(t+\Delta t)) \middle| X(t) = \mathrm{x} \right] \qquad (32)\\
&= (\Theta^{\Delta t})^{n-2}\, \mathbb{E}\left[\mathbb{E}\left[f(X(t+2\Delta t)) \middle|_{X(t+\Delta t) = X(t+\Delta t)} \right] \middle| X(t) = \mathrm{x} \right] \\
&= (\Theta^{\Delta t})^{n-2}\, \mathbb{E}\left[f(X(t+2\Delta t)) \middle| X(t) = \mathrm{x} \right] \\
&\ \vdots \\
&= \mathbb{E}\left[f(X(t+n\Delta t)) \middle| X(t) = \mathrm{x} \right] \qquad = \Theta^{n\Delta t} f(\mathrm{x})
\end{aligned}$$

2.2.1 Binomial tree model

A simple model but yet a good approximation to reality is the discrete state model. Suppose a random variable X reaches the high state HX with probability p and drops to the low state LX otherwise.



(33)

The expected value of a measuring function f is the linear combination of both scenarios. That is just how the expected value was defined in statistics.

$$\Theta f(X) := pf(HX) + (1-p)f(LX) \tag{34}$$

Of course we can build multistep trees through multiple applications of Θ. The example below shows the first two steps.

$$\begin{aligned} \Theta^2 f(X) &= \Theta\left[pf(HX) + (1-p)f(LX)\right] \\ &= p\Theta f(HX) + (1-p)\Theta f(LX) \\ &= p^2 f(HHX) + p(1-p)f(LHX) + \\ &\quad\; p(1-p)f(HLX) + (1-p)^2 f(LLX) \end{aligned} \tag{35}$$

The result simplifies in recombining trees where the operators L and H commute $LH = HL$.

2.2.2 Black-Scholes model

The Black&Scholes model is a theoretical framework for the dynamics of stock prices and is used for the valuation of stock options. Although options were the topic of the previous section on financial contracts, we will briefly discuss how the Θ operator is applied in this context. The profit that we can draw from a call or put option depends on the difference between stock price S and strike price K. In case of a call option we have the right to buy one share at price K. Our profit is consequently $S - K$. The profit is multiplied with the discount factor e^{-rt}.

$$V_{call}(S,t) = \max(S-K,0)e^{-rt} \tag{36}$$

In the notation of partial differential equations we write the change of the expected value $\Theta^M f$ in its classical form. The equation yields a unique solution for the expected option value at expiration time.

$$\frac{d}{dM}\Theta^M f = \frac{d}{dt}\Theta^M f + rS\frac{d}{dS}\Theta^M f + \frac{1}{2}\sigma^2 S^2 \frac{d^2}{dS^2}\Theta^M f \tag{37}$$

The solution of this equation is well known and can be written explicitly by a convolution with the Gaussian density. The operator Θ_{bs} solves the Black&Scholes formula for the unit time step.

$$\Theta_{bs}\, f(S,t) = \int_{\mathbb{R}} \frac{e^{-\mathrm{x}'^2/2}}{\sqrt{2\pi}} f\left(Se^{r+\sigma\mathrm{x}'-\frac{1}{2}\sigma^2}, t+1\right) d\mathrm{x}' \tag{38}$$

The expectation of the option payoff at maturity time M is computed by an application of Θ with the appropriate power. The resulting function is supplied with the initial values for S and t.

$$\Theta_{bs}^M V_{call}(S_0, 0) \tag{39}$$

Note that the operators are placed in chronological order from left to right. First, you wait M time steps, and then you evaluate the payoff. Although the operators are written left to right, their evaluation direction is opposite. First compute the value function f and then apply Θ_{bs} M times.

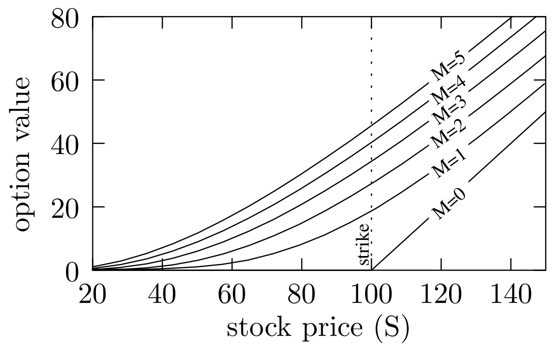


Figure 1: Call option under the Black&Scholes model. The process operator Θ_{bs} is applied M times to V_{call}

Plotted function:
$f_M(S) = {\Theta_{bs}}^M V_{call}(S, 0)$
with
$\sigma = 40\%,\ r = 5\%,\ K = 100$

The Black&Scholes model describes a stock that increases its value in expectation with the same rate as an interest rate account. The great achievement of Black and Scholes was that options prices can be computed as the expected pay off under the adjusted drift although real stock prices are experienced to grow at a significantly higher rate. Only these option prices can be reproduced without risk in a continuous buying and selling strategy called delta hedge.

2.2.3 Lévy process

The Lévy process operator is defined by a convolution with the Lévy density, that is determined by its characteristic function ϕ. The convolution kernel is extracted by a Fourier transformation of the Euler constant to the power of ϕ.

$$\Theta_L^\phi f(\mathrm{x}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{e^{\phi(y)}} f(\mathrm{x} - y) dy \tag{40}$$

The function ϕ is the logarithm of the Fourier transformed density and is uniquely determined by the Lévy triplet $[\mu, \sigma, \nu]$. Whereas μ is the drift, σ the volatility of a Brownian motion and ν is the Lévy measure reflecting the intensity of jumps of different sizes.

$$\phi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} e^{iu\mathrm{x}} - 1 - iu\mathrm{x}\, 1_{|\mathrm{x}|<1} \nu(d\mathrm{x}) \tag{41}$$

All processes that that can be homogeneously split into arbitrary time steps Δt, with $p_{\Delta t}(\mathrm{x}, \mathrm{y}) = p_{\Delta t}(\mathrm{x} - \mathrm{y})$, are Lévy processes. The most famous is the Brownian motion with $\nu = 0$.

Lemma: The super script position of ϕ to the operator Θ_L satisfies the algebraic power rule. This allows intuitive calculations with operator Θ_L and its exponent ϕ.

$$\left(\Theta_L^{\phi}\right)^n = \Theta_L^{n\phi} \tag{42}$$

Proof: The proof utilizes the fact that the convolution formula (40) turns into pointwise multiplication in Fourier space.

$$\begin{aligned}\left(\Theta_L^{\phi}\right)^n f &= \widehat{e^{\phi}} * \cdots * \widehat{e^{\phi}} * f \\ &= \widehat{e^{n\phi}} * f \qquad = \Theta_L^{n\phi} f\end{aligned} \tag{43}$$

Formula (42) also defines a default root for non integer exponents n. It is an exclusive property of Lévy processes that the convolution kernels generated by fractional powers are in fact positive probability densities.

2.3 Pricing via arbitrage

This section gives a quick introduction into arbitrage pricing to find the unique option price in a single step binomial tree. We will try to replicate the pay off of a call option precisely. In fact, there exists an investment strategy that yields the same final cash value as a call option.
In order to set up our strategy Π we write the usual operator sequence in chronological order from left to right. First we buy x stocks at price S, thus withdrawing xS from our account c. Then we by one option at a price y. Then we wait one period, sell all our stocks and exercise the option.

$$\Pi(\Theta) = \underbrace{T_c^{-\mathrm{x}S}\, T_c^{-\mathrm{y}}}_{\substack{\text{buy x stocks} \\ \text{at } S \text{ and } 1 \\ \text{option at y.}}} \underbrace{\Theta}_{\text{wait}} \underbrace{T_c^{\mathrm{x}S}}_{\substack{\text{sell} \\ \text{stocks}}} \underbrace{\left\langle \begin{array}{l} \overset{\max}{\nearrow}\ T_c^{S-K} \\ \searrow\ . \end{array}\right.}_{\text{exercise option}} \tag{44}$$

2.3.1 The process

Now we need to define what happens during the time of our inactivity. The time process consists of two effects. First, an interest rate of size rc is payed to the cash account c. And, second, two different branches are taken with probabilities p and $1-p$. In the first case S is increased by $+S$, resulting in $2S$. The other possible outcome reduces S to its half.

$$\Theta = T_c^{rc} \left\langle \begin{array}{l} \overset{p}{\nearrow}\ T_S^{+S} \\ \underset{1-p}{\searrow}\ T_S^{-\frac{1}{2}S} \end{array}\right. \tag{45}$$

Initial values The initial values of the process parameters are always inserted after the complete operator term is expanded. We will use the operator 大 to indicate this insertion. 大 is always the leftmost operator in a sequence, since the variables do no longer occur in the function after its application.

$$大 f = f\Big|_{\substack{S=100 \\ K=150} \quad \substack{c=0 \\ r=1/9}} \tag{46}$$

The operator 大 inserts the initial values and triggers a computational evaluation procedure, if necessary. Binomial tree models with short time horizons can normally be handled symbolically by computer algebra systems and thus allow the automated extraction of many implicit parameters.

2.3.2 Arbitrage-free price

In order to find the fair option price y, we have to find a hedge position x such that our final cash amount is zero under all conditions. Basically, we write our evaluation formula chronologically. First we fix the initial values, then go through our investment strategy Π and finally query our cash value c. This operator sequence evaluates the expected amount π of account c after the completion of this strategy.

$$\begin{aligned}
\pi &= 大\,\Pi(\Theta)c \qquad (47)\\
&= 大\,T_c^{-\mathrm{x}S}\,T_c^{-\mathrm{y}}\Theta T_c^{\mathrm{x}S}(c+\max(S-K,0))\\
&= 大\,T_c^{-\mathrm{x}S}\,T_c^{-\mathrm{y}}\Theta\,(c+\mathrm{x}S+\max(S-K,0))\\
&= 大\,T_c^{-\mathrm{x}S}\,T_c^{-\mathrm{y}}\big(c(1+r)+p\,(\mathrm{x}2S+\max(2S-K,0))+\\
&\quad (1-p)\,(\mathrm{x}S/2+\max(S/2-K,0))\,\big)\\
&= 大\big((c-\mathrm{x}S-\mathrm{y})(1+r)+p\,(\mathrm{x}2S+\max(2S-K,0))+\\
&\quad (1-p)\,(\mathrm{x}S/2+\max(S/2-K,0))\,\big)\\
&= p\left(50+\frac{800}{9}\,\mathrm{x}-\frac{10}{9}\,\mathrm{y}\right)+(1-p)\left(-\frac{550}{9}\,\mathrm{x}-\frac{10}{9}\,\mathrm{y}\right)
\end{aligned}$$

The final cash value is the same as has been derived in [5]. The difference is that this computation follows straight forward mathematical expansions. You should keep in mind that more realistic examples with multiple steps and additional assets quickly lead to algebraic results that can span several pages. A simple and compact calculus to command a computer algebra system is therefore essential. Fortunately, this result is short and several methods can be used to find the solution. The equation system that is to solve requires that the profit π is zero for all probabilities p.

$$\exists \mathrm{x},\mathrm{y}:\ \forall p:\pi = 0 \tag{48}$$

We can turn this into a finite system of equations, by inserting different values for p and verify the result. In this example we retrieve an option price of 55/3 and a hedge position of $-1/3$ stocks.

$$\widetilde{\mathrm{x}} = -\frac{1}{3}, \quad \widetilde{\mathrm{y}} = \frac{55}{3} \tag{49}$$

The existence of a solution is not always guaranteed. The fact that we do have a valid single solution is due to the fact that we deal with a so called complete market, in which all options can be replicated by a unique stock trading strategy.

2.3.3 Equivalent martingale measure

A simplified and efficient method for finding the same price is done with the equivalent martingale measure. There exists a pseudo probability $\widetilde{p}$ for which the discounted stock price is a martingale, i.e. the discounted expected value tomorrow is equal to todays value.

$$\exists p : \frac{\Theta S}{1+r} = S \tag{50}$$

The result is found easily.

$$\widetilde{p} = \frac{11}{27} \tag{51}$$

With the new value for p we can create a transformed process $\widetilde{\Theta}$ that evaluates the expected value under the equivalent martingale measure, where stock prices are expected to grow with the interest rate.

$$\widetilde{\Theta} = \Theta|_{p=\widetilde{p}} \tag{52}$$

This transformed operator can now be directly applied to the pay off structure of the option to compute the fair price, exactly as we did it in the Black&Scholes model (see 2.2.2).

$$大\, \frac{\widetilde{\Theta} \max(S-K,0)}{1+r} = \frac{55}{3} \tag{53}$$

The evaluation operator 大 causes the computational system to switch to a numerical scheme after the full operator term was specified.

2.4 Games

Theta-calculus is a mathematical method for the definition of activity sequences. It can be used to describe game settings in which multiple players act sequentially and optimize their situation. A brief glance into simple game settings will deepen our insight into the chronological notation of multi player settings, bargaining and information dispersion.

2.4.1 Prisoners dilemma

We will investigate a game with two players A and B, each facing a choice between cooperation (+1) and defection (−1). The selected action of the players are represented by variables a and b. The utilities U_a and U_b show how much each party desires an outcome depending on a and b.

$$U = [U_a, U_b] = [2b - a, 2a - b] \tag{54}$$

According to the utility function U, both parties will pursue a situation where the other one makes the effort (+1) while themself relaxing (−1). Clearly, each party tries to avoid the other parties preference.

$[U_a, U_b]$	$a = -1$	$a = +1$
$b = -1$	$[-1, -1]$	$[-3, +3]$
$b = +1$	$[+3, -3]$	$[+1, +1]$

For technical reasons we initialize our variables a and b to zero, such representing the positive action with T^{+1} and the negative with T^{-1}.

$$大 f = f|_{\substack{a=0\\b=0}} \tag{55}$$

2.4.2 Competition

A competitive action consists of two steps. First A acts, then B. Each party is entirely aware of the other parties preferences.

1. A decides, fully anticipating B's reaction
2. B decides, knowing A's action

The competitive game $\Pi_{\lightning}$ consists of two steps. First A optimizes her choices with respect to the first component in the utility. Then B optimizes with respect to the second component.

$$\Pi_{\lightning} = \left(\overset{\max U_1}{<} \begin{matrix} T_a^{+1} \\ \\ T_a^{-1} \end{matrix} \right) \left(\overset{\max U_2}{<} \begin{matrix} T_b^{+1} \\ \\ T_b^{-1} \end{matrix} \right) \tag{56}$$

Applying the game operator $\Pi_{\lightning}$ to U, evaluates the final utility after the game completed.

$$\begin{aligned} 大 \Pi_{\lightning} U &= 大 \max_1 \left(T_a^{+1} \max_2 \left(T_b^{+1} U, T_b^{-1} U \right), T_a^{-1} \max_2 \left(T_b^{+1} U, T_b^{-1} U \right) \right) \quad (57) \\ &= \max_1 \left(\max_2 \left([+1, +1], [-3, +3] \right), \max_2 \left([+3, -3], [-1, -1] \right) \right) \\ &= \max_1 \left([-3, +3], [-1, -1] \right) \\ &= [-1, -1] \end{aligned}$$

No matter how A decides, B will prefer abstinence (-1). Hence, A will prefer joint reluctance $([-1,-1])$ to unsupported effort $([-3,+3])$. The result is for both parties inferior to a combined action, where they could each have achieved the positive utility $([+1,+1])$. It left game theorists puzzled, why mathematically rational computation yields this inferior result.

2.4.3 Cooperation

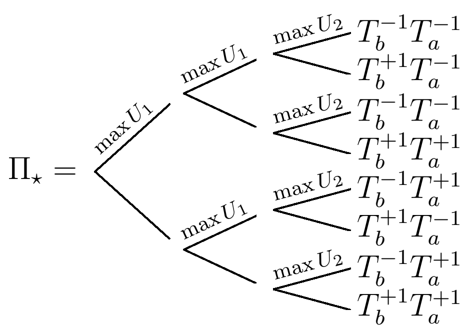
The full preliminaries for cooperative behavior were discovered by John Nash. He realized that a full disclosure of an intended reaction is essential, operationally written as a three step process.

1. A unveils the intended reaction to B's decision
2. B decides, fully informed about A's strategy
3. A sticks to the proposed strategy

Certainly, step number three presents something like a weak point. Real life contract partners will demand step two and three to occur within a very short time span and often within visual or even physical reach. Philosophers and social scientists argued in favor of various measures to ensure correspondence between step one and three, generally referred to as the social dilemma.
In step one A can indicate four different strategies, each mapping deterministically B's action onto A's reaction.

Strategy	$b \to a$	$b \to a$
1	$-1 \to -1$	$+1 \to -1$
2	$-1 \to -1$	$+1 \to +1$
3	$-1 \to +1$	$+1 \to -1$
4	$-1 \to +1$	$+1 \to +1$

The cooperative game $\Pi_\star$ now proceeds as follows. Initially, A chooses between four different strategies and optimizes with respect to the first component of the utility function. Consequently B considers her choices upon which A's reaction is fixed.



(58)

The operator notation of this game implicitly presents a formula for the evaluation of the final value for utility U.

$$\begin{aligned}
& \text{大}\,\Pi_\star U && (59)\\
= \; & \text{大} \max_1 \left(\max_2 \left(T_b^{-1} T_a^{-1} U, T_b^{+1} T_a^{-1} U \right), \max_2 \left(T_b^{-1} T_a^{-1} U, T_b^{+1} T_a^{+1} U \right), \right. \\
& \qquad \left. \max_2 \left(T_b^{-1} T_a^{+1} U, T_b^{+1} T_a^{-1} U \right), \max_2 \left(T_b^{-1} T_a^{+1} U, T_b^{+1} T_a^{+1} U \right) \right) \\
= \; & \max_1 \big(\max_2([-1,-1],[+3,-3]), \max_2([-1,-1],[+1,+1]), \\
& \qquad \max_2([-3,+3],[+3,-3]), \max_2([-3,+3],[+1,+1]) \big) \\
= \; & \max_1 \big([-1,-1],[+1,+1],[-3,+3],[-3,+3] \big) \\
= \; & [+1,+1]
\end{aligned}$$

Apparently strategy number two is preferred by player A, indicating that A is going to exactly repeat B's action. Consequently, B faces a choice between join denial and join effort. The incentive seems to go with cooperation.

2.4.4 Numéraire approach

A cooperative solution can be ensured when A's utility can be interchanged for some of B's, via some form of an exchange medium or numéraire. Assume that A and B can convert their gained utility into nominal cash values N_a and N_b, which must monotonously increase with utility U_a and U_b.

$$U_N = N_a(U_a) + N_b(U_b) \tag{60}$$

In a combined strategy A and B will exclusively maximize total wealth, expressed in a single number.

$$\Pi_N = \left(\overset{\max}{<} \begin{matrix} T_a^{+1} \\ T_a^{-1} \end{matrix} \right) \left(\overset{\max}{<} \begin{matrix} T_b^{+1} \\ T_b^{-1} \end{matrix} \right) \tag{61}$$

Applied to the equally weighted joint utility the strategy yields a maximum gain of 2, to which each party contributes +1. Cooperation is ensured.

$$\text{大}\,\Pi_N(U_a + U_b) = 2 \tag{62}$$

Other interchange functions are possible, where one party's utility makes up for a much larger cash amount. Optimal strategies can then involve mixed actions with one working (+1) and the other contemplating (−1). However a situation where both parties play −1 is never optimal and always a result of lacking cooperation.

2.5 Incomplete market

In this final example we consider a portfolio manager who periodically rebalances her hedge portfolio. Our trader has an obligation to her customers in the form of a call option with strike 10. Suppose her mission was to optimally meet her obligation in either cash or stocks with a minimum squared distance. We do not work in complete markets, since we assume trading opportunities at discrete times and will later introduce market impact. Thus all hedges will bear at least some risk. Our function f contains the final wealth and its square.

$$f = \begin{pmatrix} c + Sh - V_{call} \\ -(c + Sh - V_{call})^2 \end{pmatrix} = \begin{pmatrix} c + Sh - \max(S - 10, 0) \\ -\big(c + Sh - \max(S - 10, 0)\big)^2 \end{pmatrix} \tag{63}$$

For a least square hedge we define the utility operator U as the second component of vector f.

$$Uf = f_2 \tag{64}$$

We consider a single asset market where the price level S fluctuates according to a Brownian motion. The Lévy triplet for this process is $[\mu = 0, \sigma = 1, \nu = 0]$ and can be turned into operator form by equation (40).

$$\Theta = \Theta_L^{-\frac{1}{2}u^2} \tag{65}$$

The numeric solution to the expected final amount of cash and the expected utility is done by the numerical evaluation scheme 大. This operator evaluates an approximation to the operator term Θf and inserts the initial values for S, c and h. Depending on the chosen method 大 initializes a scenario in a Monte-Carlo simulation or retrieves the initial position in a PDE result.

$$大\Theta f = \Theta f\Big|_{\substack{S=10 \\ c=0.4 \\ h=0}} = \begin{pmatrix} 0 \\ -0.34 \end{pmatrix} \tag{66}$$

The first component tells us that the trader meets the expectation of her obligation precisely. Hence, the initial cash value of 0.4 is the expected value for the option. The low utility in the second component reveals the high risk inherent in holding the unhedged option.

2.5.1 Hedging activity

Now we want to see, if the utility can be increased by trading in the underlying stock with a reoptimization frequency of Δt. The operator that buys one stock subtracts the current stock price from cash account c and adds one stock the deposit h. The $\star$ indicates the optimal exponent.

$$\Pi(\Theta) = \left(\left(T_c^{-S}\, T_h\right)^{\star}\ \Theta^{\Delta t}\right)^{\frac{1}{\Delta t}} \tag{67}$$

According to our numeric results, the utility increases significantly with a portfolio rebalancing frequency of $\Delta t = 1/4$. The expected profit is still zero. The strategy produces no extra costs.

$$大\Pi(\Theta) f = \begin{pmatrix} 0 \\ -0.03 \end{pmatrix} \tag{68}$$

Complete market With our choice for the process Θ the risk of every obligation f can be reduced to zero by infinitely many rehedges. Markets governed by such processes are called complete markets [6].

$$\lim_{\Delta t \to 0} 大\Pi(\Theta) f = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{69}$$

2.5.2 Supply and demand

Due to the law of supply and demand real stock prices vary with the traded amount. Individual market participants will enter in the order book the prices at which they are willing to buy or sell. The more stocks we want to trade, the more people we have to satisfy and the worse is our price. We assume a linear order book in which every transaction has a price impact of κ per stock. The new strategy with $1/\Delta t$ rehedges, Π_I, considers the price impact on S.

$$\Pi_I(\Theta) = \left(\left(T_S^{\kappa} \, T_c^{-S} \, T_h \right)^{\star} \; \Theta^{\Delta t} \right)^{\frac{1}{\Delta t}} \tag{70}$$

Applied to our valuation function this reveals the expected final cash amount and the utility. The assumed market elasticity is $\kappa = 0.2$.

$$大\Pi_I(\Theta) f \begin{pmatrix} -0.05 \\ -0.04 \end{pmatrix} \tag{71}$$

Our trader is expected to loose 0.05 units of cash due to market friction. These are not transaction costs, since the selling operator is the exact inverse of the buying operator. The costs originate from procyclic trading and are gained by the anticyclic investor. If our trader wanted to reduce her loss then she had to take more risk. With respect to her quadratic utility function the presented values are optimal.

2.5.3 Market impact

Finally we might ask for the market impact of the hedging strategy on the stock price. The valuation function f is easily extended by additional components for the expected stock price S and its square.

$$大\Pi_I(\Theta) \begin{pmatrix} f \\ S \\ S^2 \end{pmatrix} = \begin{pmatrix} \Pi_I(\Theta) f \\ 10.117 \\ 103.5 \end{pmatrix} \tag{72}$$

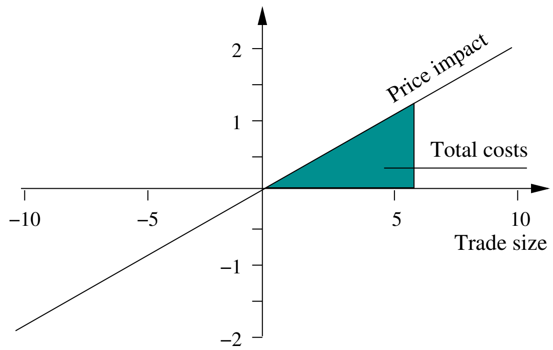


Figure 2: In the linear market impact model every traded stock shifts the price by $\kappa = 0.2$. The total cost, marked by the filled area, is proportional to the square of the trade size.

The first interesting result is that the stock price S is expected to rise by 1.17%, which is due to the fact that we bought stocks for hedging purposes but did not necessarily sell them finally. Economists will refer to this phenomena as inflation, induced by a 40 cent increase of circulating cash and a corresponding overdemand on the stock exchange. The second parameter needs some treatment to reveal its information.

$$\tilde{\sigma} = \sqrt{\mathbb{E}(S^2) - \mathbb{E}(S)^2} = \sqrt{103.5 - 10.117^2} = 1.08 \tag{73}$$

We remember that the volatility σ was initialized to one in (65) and now increased to 1.08. The final result is that our hedging strategy is procyclic and increases the market volatility of the stock by 8%. This is a realistic value for very large investments.

3 Conclusion

This document introduced an operator notation for any sequence of investment activities, game strategies and events. Three kinds of operations were considered to constitute the space of possible strategies. The first kind of occupation is inactivity. Whenever Θ occurs in an operator term it refers to a period of passive observation. During that time, external state variables may vary according to a stochastic process [6, 4] or as described by a partial differential equation [7]. The second possible activity is a transaction. The operator T initiates a deterministic effect on our parameter set. Typical instances are the transfer of goods or cash. The third and final operation models an option. Multiple operator terms $O_1 \cdots O_n$ can be offered as choices for further procedure. The decision criteria can be based on the current state and the expected values for each choice. The operator term is written in chronological order from left to right and makes use of some mathematical concepts like the operator power for repeated actions. Solutions to risk measures and expected values can be evaluated directly with either a numerical or in some instances symbolic method.

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